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EXISTENCE OF MINIMAL H-BUBBLES

PAOLO CALDIROLI
Dipartimento di Matematica, Università di Torino,
via Carlo Alberto, 10 – 10123 Torino, Italy
caldiroli@dm.unito.it

ROBERTA MUSINA
Dipartimento di Matematica ed Informatica, Università di Udine,
via delle Scienze, 206 – 33100 Udine, Italy
musina@dimi.uniud.it

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Given a function $H \in C^1(\mathbb{R}^3)$ asymptotic to a constant at infinity, we investigate the existence of H-bubbles, i.e., nontrivial, conformal surfaces parametrized by the sphere, with mean curvature H. Under some global hypotheses we prove the existence of H-bubbles with minimal energy.

1. Introduction

Since 1930, with the renowned papers by Douglas and Radó on minimal surfaces, the study of parametric two-dimensional surfaces with prescribed mean curvature, satisfying different kinds of geometrical or topological side conditions, has constituted a very challenging problem and has played a prominent role in the history of the Calculus of Variations.

Surfaces with prescribed *constant* mean curvature are usually known as "soap films" or "soap bubbles". This case has been successfully and deeply investigated by several authors, and nowadays a quite wide description of the problem is available in the literature (see the survey book by Struwe [18]).

The phenomenon of the formation of an electrified drop is closely related to soap film and soap bubbles. As experimentally observed (see for example [8], [6], [10]), an external electric field may affect the shape of the drop, and its surface curvature turns out to be nonconstant, in general.

However, as regards the mathematical treatment of the case of *nonconstant* prescribed mean curvature, only few existence results of variational type are known. Apart from few papers on the existence of a "small" solution for the Plateau problem (we quote, for instance, [10], [16] and [17], see also [5]), all the other variational-

type results hold true in a perturbative setting, namely, for curvatures of the form $H(u) = H_0 + H_1(u)$ with $H_0 \in \mathbb{R}\setminus\{0\}$ and $H_1 \in C^1(\mathbb{R}^3)\cap L^\infty$ having $\|H_1\|_\infty$ small. In particular, let us mention the papers [19], [20], [2], [12] and [13], which deal with the Plateau problem, or the corresponding Dirichlet problem.

In this paper we are interested in the existence of $\mathbb{S}^2$-type parametric surfaces in $\mathbb{R}^3$ having prescribed mean curvature H, briefly, *H-bubbles*.

More precisely, for $H \in C^1(\mathbb{R}^3)$, an H-bubble is a nonconstant conformal function $\omega\colon \mathbb{R}^2 \to \mathbb{R}^3$, smooth as a map on $\mathbb{S}^2$, satisfying the following problem:

$$\begin{cases} \Delta\omega = 2H(\omega)\omega_x \wedge \omega_y & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} |\nabla\omega|^2 < +\infty \ . \end{cases} \tag{1.1}$$

Here $\omega_x = (\frac{\partial\omega_1}{\partial x}, \frac{\partial\omega_2}{\partial x}, \frac{\partial\omega_3}{\partial x})$, $\omega_y = (\frac{\partial\omega_1}{\partial y}, \frac{\partial\omega_2}{\partial y}, \frac{\partial\omega_3}{\partial y})$, $\Delta\omega = \omega_{xx} + \omega_{yy}$, $\nabla\omega = (\omega_x, \omega_y)$, and $\wedge$ denotes the exterior product in $\mathbb{R}^3$.

In case of nonzero *constant* mean curvature $H(u) \equiv H_0$, Brezis and Coron [4] proved that the only nonconstant solutions to (1.1) are spheres of radius $|H_0|^{-1}$.

In the present paper, we study the existence of H-bubbles with minimal energy in case $H\colon \mathbb{R}^3 \to \mathbb{R}$ is a smooth function satisfying:

$(\mathbf{h_1})$ $\sup_{u\in\mathbb{R}^3} |\nabla H(u+\xi)\cdot u\, u| < 1$, for some $\xi \in \mathbb{R}^3$,

$(\mathbf{h}_\infty)$ $H(u) \to H_\infty$ as $|u| \to \infty$, for some $H_\infty \in \mathbb{R}$.

The assumption $(\mathbf{h_1})$ is a global condition on the radial component of $\nabla H(\cdot + \xi)$ that, roughly speaking, measures how far H differs from a constant.

In addition, we also need that H is nonzero on some sufficiently large set. This condition will be made clear in the following.

In order to state our result we need some preliminaries. Let us point out that problem (1.1) has a natural variational structure, since solutions to (1.1) are formally the critical points of the functional

$$\mathcal{E}_H(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + 2\int_{\mathbb{R}^2} Q(u)\cdot u_x \wedge u_y \ ,$$

where $Q\colon \mathbb{R}^3 \to \mathbb{R}^3$ is any vector field such that $\operatorname{div} Q = H$.

Roughly speaking, the functional $\int_{\mathbb{R}^2} Q(u)\cdot u_x \wedge u_y$ has the meaning of a volume, for u in a suitable space of functions. This is clear when $H(u) \equiv H_0$. Indeed in this case, taking $Q(u) = \frac{H_0}{3}u$, one deals with the standard volume functional $\int_{\mathbb{R}^2} u\cdot u_x \wedge u_y$ which is a determinant homogeneous in u and, for u constant far out, measures the algebraic volume enclosed by the surface parametrized by u. Moreover, it turns out to be bounded with respect to the Dirichlet integral by the Bononcini-Wente isoperimetric inequality.

These facts hold true more generally when H is a bounded nonzero function on $\mathbb{R}^3$ (see [16]). In particular, the functional $\int_{\mathbb{R}^2} Q(u)\cdot u_x \wedge u_y$ is essentially cubic in u and it satisfies a generalized isoperimetric inequality. For this reason, we expect

that $\mathcal{E}_H$ has a mountain pass structure, and this gives an indication for the existence of a nontrivial critical point.

The natural space in order to look for $\mathbb{S}^2$-type solutions seems to be the Sobolev space $H^1(\mathbb{S}^2, \mathbb{R}^3)$, modulo stereographic projection. However, working with this space gives some technical difficulties due to the fact that H may be nonconstant. In any case, we can define a mountain pass level for $\mathcal{E}_H$ restricted to some class of smooth functions. In addition, thanks to the assumption $(\mathbf{h_1})$, we can restrict ourselves to radial paths spanned by functions in $\mathcal{S}_\xi = \{\xi + C_c^\infty(\mathbb{R}^2, \mathbb{R}^3) : u \not\equiv \xi\}$, where $\xi \in \mathbb{R}^3$ is the same as in $(\mathbf{h_1})$. Thus we are lead to introduce the value

$$c_H = \inf_{u \in \mathcal{S}_\xi} \sup_{s>0} \mathcal{E}_H(su) .$$

The assumption $(\mathbf{h}_\infty)$ guarantees that

$$0 < c_H \leq \frac{4\pi}{3H_\infty^2} .$$

Note that if $H_\infty \neq 0$, the value $\frac{4\pi}{3H_\infty^2}$ equals the mountain pass level for the energy functional $\mathcal{E}_{H_\infty}$ corresponding to the constant mean curvature H_∞. Moreover by the results proved by Brezis and Coron in [4], this value is the least critical value for $\mathcal{E}_{H_\infty}$ in $H^1(\mathbb{S}^2, \mathbb{R}^3)$, and it is attained by the spheres (with degree 1) of radius $|H_\infty|^{-1}$. Now, our result can be stated as follows:

Theorem 1.1 *Let $H \in C^1(\mathbb{R}^3)$ satisfy $(\mathbf{h_1})$ and $(\mathbf{h}_\infty)$. If*

$$(*) \qquad c_H < \frac{4\pi}{3H_\infty^2}$$

holds, then there exists an H-bubble ω such that $\mathcal{E}_H(\omega) = c_H$. Moreover, called $\mathcal{B}_H$ the set of H-bubbles, it holds that $c_H = \inf_{\omega \in \mathcal{B}_H} \mathcal{E}_H(\omega)$.

We point out that, thanks to $(\mathbf{h_1})$, the condition $(*)$ requires that $\mathcal{E}_H(\bar{u}) < 0$ for some $\bar{u} \in \mathcal{S}_\xi$ and then excluded the case $H \equiv 0$. Clearly, when $\mathcal{E}_H(\bar{u}) < 0$ somewhere and $H_\infty = 0$, then $(*)$ is automatically satisfied. Moreover, when $H_\infty > 0$, the condition $(*)$ turns out to be true if $H(u) > H_\infty$ for $|u|$ large. Note that, in general, even if $H(u) = H_\infty$ for $|u| \geq R$, Theorem 1.1 ensures that the H-bubble we find is different from the H_∞-bubble located in the region $|u| \geq R$.

We also notice that in general we have no information about the position of the H-bubble given by Theorem 1.1. In particular, we can exhibit examples of radial curvatures H for which H-bubbles with minimal energy exist but cannot be radial.

The main difficulties in approaching problem (1.1) with variational methods concern the study of the Palais-Smale sequences. In particular, we emphasize the following problems: boundedness of a Palais-Smale sequence with respect to the Dirichlet norm, and in L^∞; blow up analysis for a (bounded) Palais-Smale sequence. Concerning the first problem, the assumption $(\mathbf{h_1})$ can be useful in order to guarantee the boundedness with respect to the gradient L^2-norm. However the

boundedness in L^∞ in general cannot be deduced *a priori* and it is not just a technical difficulty. In fact, one can exhibit examples of Palais-Smale sequences which are bounded with respect to the Dirichlet norm, but not in L^∞, and the lack of boundedness in L^∞ cannot be eliminated in any way.

Hence, because of these difficulties, we tackle the problem by using an approximation method in the spirit of a celebrated paper by Sacks and Uhlenbeck [15]. More precisely, we construct a family of approximating solutions on which global and local estimates can be proved. In particular, assuming that H is constant far out, we can obtain boundedness both with respect to the Dirichlet norm, and in L^∞. Then, a limit procedure, involving a (partial) blow up analysis, is carried out, in order to show the existence of an H-bubble with minimal energy. In the last step, we remove the assumption that H is constant far out, by an approximation argument on the curvature function, and we recover the full result stated in Theorem 1.1.

We point out that for a curvature $H \in C^1(\mathbb{R}^3)$ satisfying $(\mathbf{h_1})$ and such that $H(u) \equiv H_\infty \neq 0$ for $|u|$ large, the set $\mathcal{B}_H$ of H-bubbles is nonempty *a priori*, and the existence of a minimal H-bubble can be obtained with a direct argument, just minimizing the energy functional $\mathcal{E}_H$ over $\mathcal{B}_H$, without using the above mentioned approximation method. In fact, the hard step lies in removing the condition that H is constant far out, just asking to H the asymptotic behaviour stated in $(\mathbf{h}_\infty)$. To this goal, it is important to know that the energy of the minimal H-bubble is exactly c_H, and proving this needs either a sharp study of the behaviour of the Palais Smale sequences, or an (almost equivalent) approximation argument as, for instance, the Sacks-Uhlenbeck type argument that we develop. This step requires much more work and constitutes the largest part of this paper.

We finally mention a result by Bethuel and Rey [2] that states the existence of an H-bubble passing through an arbitrarily prescribed point in $\mathbb{R}^3$ in case H is a perturbation of a nonzero constant. This result expresses the fact that the bubbles with constant curvature $H_0 \neq 0$ are stable with respect to small L^∞ perturbations of H_0. Actually, in our opinion, the proof of this result is not completely clear and we are not able to recover it with our method.

In fact, we think that the problem of existence of H-bubbles for a prescribed bounded curvature function H has some similarities with a semilinear elliptic problem on $\mathbb{R}^N$ of the form

$$\begin{cases} -\Delta u + u = a(x)u^p & \text{on } \mathbb{R}^N \\ u > 0 & \text{on } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.2}$$

where $1 < p < \frac{N+2}{N-2}$ and a is a bounded positive function on $\mathbb{R}^N$. It is known that the existence of solutions to (1.2) is strongly affected by the behaviour of the coefficient $a(x)$, and in some cases problem (1.2) has no solution. In particular, this may happen also when $a(x)$ is a small L^∞ perturbation of a positive constant.

In our opinion, similar considerations hold also for the problem of H-bubbles, and the behaviour of $H(u)$ plays a similar role of the coefficient $a(x)$ in (1.2). Hence, as well as for problem (1.2), we suspect that the existence of H-bubbles with minimal energy may depend in a very sensitive way on the function H.

2. The variational approach

This Section is structured as follows. In the first part we introduce some notation in view of setting up a variational framework to study problem (1.1). In particular we define the H-volume functional, the energy functional associated to problem (1.1), and we recall some generalized isoperimetric inequality. In the second part we define a mountain pass level c_H for the energy functional $\mathcal{E}_H$ and we discuss some properties related to the value c_H strongly depending on the assumption $(\mathbf{h_1})$.

2.1. *Notation and isoperimetric inequality*

First, let us introduce the space

$$X = \{v \circ \phi : v \in H^1(\mathbb{S}^2, \mathbb{R}^3)\}$$

where $\phi: \mathbb{R}^2 \to \mathbb{S}^2$ is the (inverse of the) standard stereographic projection and it is given by

$$\phi(z) = (\mu x, \mu y, 1 - \mu) \ , \quad \mu = \mu(z) = \frac{2}{1 + |z|^2} \ , \tag{2.1}$$

being $z = (x, y)$ and $|z|^2 = x^2 + y^2$. Notice that $u \in X$ if and only if $u, \hat{u} \in H^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$ and $\int_{\mathbb{R}^2} |\nabla u|^2 < +\infty$, where $\hat{u}(z) = u\big(\frac{z}{|z|^2}\big)$. Let us also set $H^1_0 = H^1_0(D, \mathbb{R}^3)$, where D is the open unit disc in $\mathbb{R}^2$. Clearly, $H^1_0 \subset X$. For every $u \in X$ we denote the Dirichlet integral by

$$\mathcal{D}(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \ .$$

Now, given $H \in C^1(\mathbb{R}^3)$, we construct the H-volume functional as follows. Set

$$m_H(u) = \int_0^1 H(su)s^2 \, ds \ .$$

Thus, for every $u \in \mathbb{R}^3$ one has

$$\operatorname{div}(m_H(u)u) = H(u) \ . \tag{2.2}$$

Then, let $\mathcal{V}_H: X \cap L^\infty \to \mathbb{R}$ be defined by

$$\mathcal{V}_H(u) = \int_{\mathbb{R}^2} m_H(u) u \cdot u_x \wedge u_y \ .$$

In case $H(u) \equiv 1$, one has $m_H(u) \equiv \frac{1}{3}$, and the functional $\mathcal{V}_H$ reduces to the classical volume functional which satisfies the standard isoperimetric inequality. In fact the following generalization holds, as proved by Steffen in [16].

Lemma 2.1 *If $H \in C^1(\mathbb{R}^3)$ is bounded on $\mathbb{R}^3$ then there exists $S_H > 0$ such that*

$$S_H |\mathcal{V}_H(u)|^{2/3} \le \mathcal{D}(u) \quad \text{for every } u \in X \cap L^\infty \ . \tag{2.3}$$

Remark 2.2 In fact Steffen in [16] proves that the functional $\mathcal{V}_H$ admits a continuous extension on H_0^1 and (2.3) holds true also for every $u \in H_0^1$.

Finally we introduce the energy functional $\mathcal{E}_H : X \cap L^\infty \to \mathbb{R}$, defined for every $u \in X \cap L^\infty$ by

$$\mathcal{E}_H(u) = \mathcal{D}(u) + 2\mathcal{V}_H(u) \ .$$

In the following result we state some properties of the functional $\mathcal{E}_H$.

Lemma 2.3 *Let $H \in C^1(\mathbb{R}^3)$. Then:*

(i) *for every $u \in X \cap L^\infty$ one has $\mathcal{E}_H(su) = s^2\mathcal{D}(u) + o(s^2)$ as $s \to 0$,*

(ii) *for every $u \in X \cap L^\infty$ and for $h \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ the directional derivative of $\mathcal{E}_H$ at u along h exists, and it is given by*

$$d\mathcal{E}_H(u)h = \int_{\mathbb{R}^2} \nabla u \cdot \nabla h + 2 \int_{\mathbb{R}^2} H(u) h \cdot u_x \wedge u_y \ ,$$

(iii) *for every bounded solution ω to (1.1) one has*

$$\mathcal{D}(\omega) + \int_{\mathbb{R}^2} H(\omega)\omega \cdot \omega_x \wedge \omega_y = 0 \ . \tag{2.4}$$

Remark 2.4 If $\omega \in X \cap L^\infty$ is a weak solution to (1.1), i.e., $d\mathcal{E}_H(\omega)h = 0$ for every $h \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$, then, since $H \in C^1(\mathbb{R}^3)$, by a Heinz regularity result [9], $\omega \in C^3(\mathbb{R}^2, \mathbb{R}^3)$, it is conformal, and smooth as a map on $\mathbb{S}^2$. In particular there exists $\lim_{|z|\to\infty} \omega(z) = \omega_\infty \in \mathbb{R}^3$.

Proof. Part (i) is a consequence of Lemma 2.1. Part (ii) follows by the results in [11], using (2.2). Finally, (2.4) can be proved multiplying the system $\Delta\omega = 2H(\omega)\omega_x \wedge \omega_y$ by ω, integrating on D_R, and passing to the limit as $R \to +\infty$. □

To conclude this Subsection, we point out a consequence of assumption ($\mathbf{h}_\infty$). Actually, the following result holds true under a much weaker condition.

Lemma 2.5 *Let $H \in C^1(\mathbb{R}^3, \mathbb{R})$ satisfy*

$$|H(su)| \ge H_0 > 0 \quad \text{for } s \ge s_0 \text{ and } u \in \Sigma \ , \tag{2.5}$$

being Σ a nonempty open set in $\mathbb{S}^2$. Then there exists $\bar{u} \in H_0^1 \cap L^\infty$ such that $\mathcal{E}_H(s\bar{u}) \to -\infty$ as $s \to +\infty$.

Proof. Thanks to the rotational invariance of the problem we may assume that Σ is an open neighborhood of the point $-e_3 = (0, 0, -1)$. Furthermore, let us suppose that $H(su) \ge H_0 > 0$ for $s > s_0$ and $u \in \Sigma$. For $\delta \in (0, 1)$ let us define

$$u^\delta(z) = \begin{cases} \phi(z) & \text{as } |z| < \delta \\ \frac{1-|z|}{1-\delta} \ \phi\left(\frac{\delta}{|z|} z\right) & \text{as } \delta \le |z| \le 1 \ , \end{cases}$$

where $\phi\colon \mathbb{R}^2 \to \mathbb{S}^2$ is the function introduced in (2.1). It holds that $u^\delta \in H^1_0 \cap L^\infty$, and u^δ parametrizes the boundary of the sector of cone defined by

$$A_\delta = \{\xi \in \mathbb{R}^3 : -|\xi| \cos\theta_\delta > \xi \cdot e_3 \ , \ |\xi| < 1\}$$

where $\theta_\delta = \arccos \frac{1-\delta^2}{1+\delta^2}$. In addition one has that

$$u^\delta(z) \cdot u^\delta_x(z) \wedge u^\delta_y(z) = \begin{cases} -\mu(z)^2 & \text{as } |z| < \delta \\ 0 & \text{as } |z| > \delta \end{cases}$$

and for every $s > 0$, by the divergence theorem,

$$\mathcal{V}_H(su^\delta) = -\int_{sA_\delta} H(\xi)\; d\xi \ .$$

Since $\phi(0) = -e_3$ and ϕ is continuous, we can find $\delta_0 \in (0,1)$ such that $\phi(z) \in \Sigma$ as $|z| < \delta_0$. Set $\bar{u} = u^{\delta_0}$ and $A = A_{\delta_0}$. Therefore, by the hypothesis, for $s > s_0$ one has

$$\begin{aligned} \mathcal{V}_H(s\bar{u}) &= -\int_{s_0 A} H(\xi)\; d\xi - \int_{sA\setminus s_0 A} H(\xi)\; d\xi \\ &\le \mathcal{V}_H(s_0\bar{u}) - \int_{sA\setminus s_0 A} H_0\; d\xi \\ &= \mathcal{V}_H(s_0\bar{u}) - \mathcal{V}_{H_0}(s_0\bar{u}) - s^3|\mathcal{V}_{H_0}(\bar{u})| \ . \end{aligned}$$

Then

$$\mathcal{E}_H(s\bar{u}) \le s^2\mathcal{D}(\bar{u}) + 2(\mathcal{V}_H(s_0\bar{u}) - \mathcal{V}_{H_0}(s_0\bar{u})) - 2s^3|\mathcal{V}_{H_0}(\bar{u})| \ .$$

Passing to the limit as $s \to +\infty$ we obtain the thesis. Finally, we observe that in case $H(su) \le H_0 < 0$ for $s > s_0$ and $u \in \Sigma$, one can repeat the same argument taking $v(x,y) = u(y,x)$. □

2.2. *The mountain pass level*

Assume that $H \in C^1(\mathbb{R}^3) \cap L^\infty$ is such that there exists $\bar{u} \in C^\infty_c(\mathbb{R}^2, \mathbb{R}^3)$ with $\mathcal{E}_H(\bar{u}) < 0$. In particular, this excludes the case $H \equiv 0$. Then, let

$$c_H = \inf_{\substack{u \in C^\infty_c(\mathbb{R}^2,\mathbb{R}^3) \\ u \ne 0}} \ \sup_{s>0} \ \mathcal{E}_H(su) \ . \tag{2.6}$$

Note that c_H is well defined and, thanks to Lemma 2.1, it is positive and finite. In particular, by (2.3), one can estimate

$$c_H \ge \left(\frac{S_H}{3}\right)^3 ,$$

where S_H is the isoperimetric constant associated to H.

Remark 2.6 When $H(u) \equiv H_0 \in \mathbb{R} \setminus \{0\}$, the volume functional is purely cubic and one can easily prove that

$$c_{H_0} = \left(\frac{S_{H_0}}{3}\right)^3 = \frac{4\pi}{3H_0^2} = \mathcal{E}_{H_0}(\omega^0) = \sup_{s>0} \mathcal{E}_{H_0}(s\omega^0)$$

where $\omega^0 = \frac{1}{H_0}\phi$ and ϕ is defined in (2.1). Notice that ω^0 is a conformal parametrization of the sphere of radius $|H_0|^{-1}$ centered at the origin, it satisfies $\Delta\omega^0 = 2H_0\omega_x^0 \wedge \omega_y^0$ on $\mathbb{R}^2$, $\mathcal{D}(\omega^0) = \frac{4\pi}{H_0^2}$, and $\mathcal{V}_{H_0}(\omega^0) = -\frac{4\pi}{3H_0^2}$.

The results that follow better explain the role of the condition $(\mathbf{h_1})$ with respect to the definition of c_H. To this extent, we point out that, since problem (1.1) is invariant under translations, in the assumption $(\mathbf{h_1})$ we may suppose that $\xi = 0$. Hence, setting:

$$M_H = \sup_{u\in\mathbb{R}^3} |\nabla H(u) \cdot u\ u| \tag{2.7}$$

the hypothesis $(\mathbf{h_1})$ reads: $M_H < 1$. It is convenient to introduce also the value

$$\bar{M}_H = 2 \sup_{u\in\mathbb{R}^3} |(H(u) - 3m_H(u))u|\ . \tag{2.8}$$

In fact, several estimates in the sequel need a bound just on $\bar{M}_H$.

Remark 2.7 (i) By (2.2) and by the definition of m_H, it turns out that $\bar{M}_H \le M_H$, but the strict inequality may also occur. Indeed one can construct functions $H \in C^1(\mathbb{R}^3)$ such that $M_H = +\infty$, while $\bar{M}_H < +\infty$.
(ii) If $H \in C^1(\mathbb{R}^3)$ satisfies $\bar{M}_H < +\infty$ then it turns out that $H \in L^\infty(\mathbb{R}^3)$. Furthermore, for every $u \in \mathbb{S}^2$ there exists $\lim_{s\to+\infty} H(su) = \hat{H}(u) \in \mathbb{R}$ and $\hat{H} \in C^0(\mathbb{S}^2)$. Thus, if $\bar{M}_H < +\infty$, then the condition (2.5) used in Lemma 2.5 is verified whenever $\limsup_{s\to+\infty} |H(su)| > 0$ for some $u \in \mathbb{S}^2$.

First, we give a positive lower bound on the energy of any H-bubble.

Proposition 2.8 *Let $H \in C^1(\mathbb{R}^3)$ satisfy $(\mathbf{h_1})$. If ω is an H-bubble, then $\mathcal{E}_H(\omega) \ge c_H$.*

The proof of Proposition 2.8 is based on the following Lemma.

Lemma 2.9 *Let $H \in C^1(\mathbb{R}^3)$ satisfy $\bar{M}_H < 1$ and let $u \in H_0^1 \cap L^\infty \setminus \{0\}$.*

(i) *If $\sup_{s>0} \mathcal{E}_H(su) < +\infty$ then $\mathcal{E}_H(su) \le as^2 - bs^3$ for every $s > 0$, with $a, b > 0$ depending on u,*

(ii) *if $\mathcal{E}_H(s_0u) < 0$ for some $s_0 > 0$ then $\sup_{s>0} \mathcal{E}_H(su) = \max_{s\in[0,s_0]} \mathcal{E}_H(su)$,*

(iii) *if $\sup_{s>0} \mathcal{E}_H(su) = \mathcal{E}_H(\bar{s}u)$, then $\mathcal{V}_H(\bar{s}u) < 0$.*

Proof. Fix $u \in H_0^1 \cap L^\infty \setminus \{0\}$ and set $f(s) = \mathcal{E}_H(su)$ for every $s \ge 0$. Notice that f is differentiable and

$$f'(s) = s\int_D |\nabla u|^2 + 2s^2 \int_D H(su)u \cdot u_x \wedge u_y\ .$$

Using (2.8), one has that

$$f'(s) \leq -(1-\bar{M}_H)\mathcal{D}(u)s + \frac{3}{s} f(s) \ . \tag{2.9}$$

If $\sup_{s>0} f(s) < +\infty$, since $\bar{M}_H < 1$, (2.9) implies that $\lim_{s\to+\infty} f'(s) = -\infty$ and then there exists $s_0 > 0$ such that $f(s) < 0$ for $s \geq s_0$. Setting $\bar{a} = (1-\bar{M}_H)\mathcal{D}(u)$ and integrating (2.9) over $[s_0, s]$ one obtains

$$f(s) \leq \left(\frac{f(s_0)}{s_0^3} - \frac{\bar{a}}{s_0}\right) s^3 + \bar{a}s^2 \tag{2.10}$$

for every $s \geq s_0$. Keeping into account that $f(s) = s^2\mathcal{D}(u) + o(s^2)$ as $s \to 0^+$, one can find $a \geq \bar{a}$ such that

$$f(s) \leq \left(\frac{f(s_0)}{s_0^3} - \frac{\bar{a}}{s_0}\right) s^3 + as^2$$

for every $s \geq 0$, namely (i). Now, let us prove (ii). If $f(s_0) < 0$, by (2.10), one infers that $\sup_{s>0} f(s) < +\infty$. Moreover (2.9) implies in particular that $f'(s) < 0$ whenever $f(s) \leq 0$. Hence also (ii) holds true. Finally, if $\sup_{s>0} \mathcal{E}_H(su) = \mathcal{E}_H(\bar{s}u)$, then $f'(\bar{s}) = 0$, and consequently, by (2.8),

$$3\mathcal{V}_H(\bar{s}u) = 3\mathcal{V}_H(\bar{s}u) - \bar{s}f'(\bar{s}) \leq -\bar{s}^2\left(1 - \frac{\bar{M}_H}{2}\right)\mathcal{D}(u) < 0 \ ,$$

that is (iii). □

Proof of Proposition 2.8. By Remark 2.4 an H-bubble ω is smooth and bounded. Moreover the mapping $f(s) = \mathcal{E}_H(s\omega)$ is well defined, and twice differentiable on $(0, +\infty)$, with

$$f''(s) = \int_D |\nabla\omega|^2 + 4s\int_D H(s\omega)\omega \cdot \omega_x \wedge \omega_y + 2s^2 \int_D \nabla H(s\omega) \cdot \omega \ \omega \cdot \omega_x \wedge \omega_y \ .$$

Since $M_H < 1$, one obtains that

$$f''(s) \leq -2(1-M_H)\mathcal{D}(u) + \frac{2}{s} f'(s) \ . \tag{2.11}$$

In particular, by (2.11), if $f'(\bar{s}) = 0$ for some $\bar{s} > 0$ then $f''(\bar{s}) < 0$. This shows that there exists at most one value $\bar{s} > 0$ where $f'(\bar{s}) = 0$. In fact, one knows that $f'(1) = 0$ because of (2.4). Hence $\sup_{s>0} f(s) = f(1)$ and, arguing as in the proof of Lemma 2.9, $f(s) \to -\infty$ as $s \to +\infty$. Now, for every $\delta \in (0,1)$ let $u^\delta : D \to \mathbb{R}^3$ be defined as follows:

$$u^\delta(z) = \begin{cases} 0 & \text{as } |z| \geq \delta \\ \left(\frac{\log|z|}{\log\delta} - 1\right)\omega_\infty & \text{as } \delta^2 \leq |z| < \delta \\ \left(\frac{\log|z|}{2\log\delta} - 1\right)(\omega^\delta(z) - \omega_\infty) + \omega_\infty & \text{as } \delta^4 \leq |z| < \delta^2 \\ \omega^\delta(z) & \text{as } |z| < \delta^4 \end{cases}$$

where $\omega_\infty = \lim_{|z|\to\infty}\omega(z)$, and $\omega^\delta(z) = \omega(\frac{z}{\delta^5})$. Note that $u^\delta \in H^1_0 \cap L^\infty$ and $\|u^\delta\|_\infty \le \|\omega\|_\infty$. Let us set $f_\delta(s) = \mathcal{E}_H(su^\delta)$. We claim that for every $s' > 0$

$$\sup_{s\in[0,s']} |f_\delta(s) - f(s)| \to 0 \quad \text{as } \delta \to 0 \ . \tag{2.12}$$

Assuming for a moment that (2.12) holds, let us complete the proof. Let $s_0 > 1$ be such that $f(s_0) < 0$. By (2.12), for $\delta > 0$ small enough, $f_\delta(s_0) < 0$ and then, by Lemma 2.9, $\sup_{s>0} f_\delta(s)$ is attained in $(0, s_0)$. Hence, using again (2.12), we have

$$c_H \le \sup_{s>0} f_\delta(s) = \max_{s\in[0,s_0]} f_\delta(s) \le \max_{s\in[0,s_0]} f(s) + o(1) = f(1) + o(1) \ .$$

Therefore the thesis follows. Finally, let us prove the claim (2.12). For every $s \ge 0$ we can write

$$\begin{aligned} f_\delta(s) - f(s) \quad &= \quad s^2 \left(\int_{|z|>\delta^4} |\nabla u^\delta|^2 - \int_{|z|>\delta^{-1}} |\nabla\omega|^2 \right) \\ &+ \quad 2s^3 \left(\int_{|z|>\delta^4} m_H(su^\delta)u^\delta \cdot u^\delta_x \wedge u^\delta_y - \int_{|z|>\delta^{-1}} m_H(s\omega)\omega \cdot \omega_x \wedge \omega_y \right). \end{aligned}$$

We observe that

$$\begin{aligned} 2\left| \int_{|z|>\delta^4} m_H(su^\delta)u^\delta \cdot u^\delta_x \wedge u^\delta_y \right| \quad &\le \quad \|m_H\|_\infty \|\omega\|_\infty \int_{|z|>\delta^4} |\nabla u^\delta|^2 \\ 2\left| \int_{|z|>\delta^{-1}} m_H(s\omega)\omega \cdot \omega_x \wedge \omega_y \right| \quad &\le \quad \|m_H\|_\infty \|\omega\|_\infty \int_{|z|>\delta^{-1}} |\nabla \omega|^2. \end{aligned}$$

Moreover, one can check that

$$\int_{|z|>\delta^4} |\nabla u^\delta|^2 \to 0 \quad \text{as } \delta \to 0 \ ,$$

and, since $\omega \in X$, also

$$\int_{|z|>\delta^{-1}} |\nabla \omega|^2 \to 0 \quad \text{as } \delta \to 0 \ .$$

Therefore (2.12) immediately follows and this concludes the proof. □

Notice that the full condition $M_H < 1$ enters just in the previous step. Now we are going to prove two technical Lemmata that will be used in the sequel.

Lemma 2.10 *Let $H \in C^1(\mathbb{R}^3)$ satisfy $\bar{M}_H < 1$. Then $c_H \le c_{\lambda H}$ for every $\lambda \in (0, 1]$.*

Proof. Firstly, notice that for $\lambda \in (0, 1]$, the isoperimetric inequality (2.3) holds true also for λH (with $S_{\lambda H} = \lambda^{-\frac{2}{3}} S_H$), and then the value $c_{\lambda H}$ is well defined

and positive. Suppose that it is finite and, given $\epsilon > 0$, let $u \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3) \setminus \{0\}$ be such that $\sup_{s>0} \mathcal{E}_{\lambda H}(su) < c_{\lambda H} + \epsilon$. Since $\bar{M}_{\lambda H} = \lambda \bar{M}_H < 1$, by Lemma 2.9, $\lim_{s\to+\infty} \mathcal{E}_{\lambda H}(su) = -\infty$. In particular, $\mathcal{V}_{\lambda H}(su) < 0$ for s large. Hence $\mathcal{E}_H(su) \le \mathcal{E}_{\lambda H}(su) < 0$ for s large. Using again Lemma 2.9 there exists $\bar{s} > 0$ such that $\sup_{s>0} \mathcal{E}_H(su) = \mathcal{E}_H(\bar{s}u)$. Furthermore $\mathcal{V}_H(\bar{s}u) < 0$. Therefore

$$c_H \le \mathcal{E}_H(\bar{s}u) = \mathcal{E}_{\lambda H}(\bar{s}u) + 2(1-\lambda)\mathcal{V}_H(\bar{s}u) \le \mathcal{E}_{\lambda H}(\bar{s}u) \le \sup_{s>0} \mathcal{E}_{\lambda H}(su) \le c_{\lambda H} + \epsilon \ .$$

Then the thesis follows because of the arbitrariness of $\epsilon > 0$. □

The next result states the upper semicontinuity of c_H with respect to H.

Lemma 2.11 *Let $H \in C^1(\mathbb{R}^3)$ satisfy $\bar{M}_H < 1$. Let $(H_n) \subset C^1(\mathbb{R}^3)$ be a sequence of functions satisfying $\bar{M}_{H_n} < 1$, and such that $H_n \to H$ uniformly on compact sets of $\mathbb{R}^3$. Then $\limsup_{n\to+\infty} c_{H_n} \le c_H$.*

Proof. Suppose that c_H is finite and, given $\epsilon > 0$ take $u \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3) \setminus \{0\}$ such that $\sup_{s>0} \mathcal{E}_H(su) < c_H + \epsilon$. One can check that $\lim_{n\to+\infty} \mathcal{E}_{H_n}(su) = \mathcal{E}_H(su)$ for every $s \ge 0$. By Lemma 2.9, $\mathcal{E}_H(s_0 u) < 0$ for some $s_0 > 0$, and then also $\mathcal{E}_{H_n}(s_0 u) < 0$ for $n \in \mathbb{N}$ large enough. Therefore, since H_n satisfies $\bar{M}_{H_n} < 1$, using again Lemma 2.9, $\sup_{s>0} \mathcal{E}_{H_n}(su) = \mathcal{E}_{H_n}(\bar{s}_n u)$ for some $\bar{s}_n \in [0, s_0]$. Then, for a subsequence, $\bar{s}_n \to \bar{s}$ and, since $H_n \to H$ uniformly on compact sets, $\mathcal{E}_{H_n}(\bar{s}_n u) \to \mathcal{E}_H(\bar{s}u)$. Consequently one has

$$c_{H_n} \le \mathcal{E}_{H_n}(\bar{s}_n u) = \mathcal{E}_H(\bar{s}u) + o(1) \le \sup_{s>0} \mathcal{E}_H(su) + o(1) \le c_H + \epsilon + o(1).$$

Passing to the limit as $n \to +\infty$ and taking into account of the arbitrariness of $\epsilon > 0$, the thesis is proved. □

Lastly, we give an estimate for c_H from above. Here, just the assumption $(\mathbf{h}_\infty)$, and in fact a more general condition, is enough.

Lemma 2.12 *Let $H \in C^1(\mathbb{R}^3)$ satisfy* (2.5) *for some nonempty open set $\Sigma \subset \mathbb{S}^2$. Then $c_H \le \frac{4\pi}{3H_0^2}$.*

Proof. As in the proof of Lemma 2.5, we may assume that Σ is an open neighborhood of the point $-e_3 = (0,0,-1)$ and that $H(su) \ge H_0 > 0$ for $s > s_0$ and $u \in \Sigma$. Let us consider the function $\omega^0 \colon \mathbb{R}^2 \to \mathbb{R}^3$ defined as in Remark 2.6. For every $r > 0$ set $\omega^r = \omega^0 - re_3$. Notice that ω^r is a conformal parametrization of a sphere of radius $r_0 = \frac{1}{H_0}$ and center $-re_3$. Hence, using the divergence theorem, one has that

$$\mathcal{V}_H(s\omega^r) = -\int_{B_{sr_0}(-se_3)} H(\xi)\ d\xi = -s^3 \int_{B_{r_0}(0)} H(s\xi - se_3)\ d\xi \ . \tag{2.13}$$

Setting $s_r = \frac{s_0}{r - r_0}$ and using (2.13), one obtains that for $s \in [0, s_r]$

$$\mathcal{E}_H(s\omega^r) \le 4\pi(r_0 s_r)^2 + \frac{8\pi}{3}\|H\|_\infty (r_0 s_r)^3 = O(s_r^2) \ ,$$

while, for $s \geq s_r$, by the hypothesis (2.5), one has

$$\mathcal{E}_H(s\omega^r) \leq 4\pi(r_0 s)^2 - \frac{8\pi}{3} H_0 (r_0 s)^3 \leq \frac{4\pi}{3H_0^2} \ .$$

Then

$$\sup_{s>0} \mathcal{E}_H(s\omega^r) \leq \max\left\{\frac{4\pi}{3H_0^2}, O(s_r^2)\right\} = \frac{4\pi}{3H_0^2} \tag{2.14}$$

for $r > 0$ large enough. Now, as in the proof of Proposition 2.8, one can construct $u^{r,\delta} \in H_0^1 \cap L^\infty$ such that $\sup_{s>0} \mathcal{E}_H(su^{r,\delta}) \leq \sup_{s>0} \mathcal{E}_H(s\omega^r) + o(1)$, with $o(1) \to 0$ as $\delta \to 0$. Hence, by (2.14), one obtains $c_H \leq \frac{4\pi}{3H_0^2} + o(1)$, that is, the thesis. □

From the previous proof, one immediately infers the next estimate.

Corollary 2.13 *Let $H \in C^1(\mathbb{R}^3)$ satisfy $(\mathbf{h}_\infty)$. Then $c_H \leq \frac{4\pi}{3H_\infty^2}$. If, in addition, $H(u) > H_\infty > 0$ for $|u|$ large, then $c_H < \frac{4\pi}{3H_\infty^2}$.*

3. Approximating problems

Aim of this Section is to introduce a family of perturbed energy functionals having a mountain pass critical point at a level which approximate the value c_H introduced in the previous Section.

The advantage in following this procedure (already used in a different framework by Sacks and Uhlenbeck [15]) is due to the possibility to obtain some uniform global and local estimates on the critical points of the perturbed problems.

Thus, for every $\alpha > 1$ (α will be taken close to 1) we consider the Sobolev space $H_0^{1,2\alpha} = H_0^{1,2\alpha}(D, \mathbb{R}^2)$ and the functional $\mathcal{E}_H^\alpha \colon H_0^{1,2\alpha} \to \mathbb{R}$ defined by

$$\mathcal{E}_H^\alpha(u) = \frac{1}{2\alpha} \int_D \big((1 + |\nabla u|^2)^\alpha - 1\big) + 2\mathcal{V}_H(u) \ .$$

It is convenient to denote

$$\mathcal{D}^\alpha(u) = \frac{1}{2\alpha} \int_D \big((1 + |\nabla u|^2)^\alpha - 1\big) \ .$$

Since $H^{1,2\alpha} \hookrightarrow L^\infty \cap H^1$, the functional $\mathcal{E}_H^\alpha$ turns out to be well defined and regular on $H_0^{1,2\alpha}$, when H is any bounded, smooth function. More precisely, $\mathcal{E}_H^\alpha$ is of class C^1 on $H_0^{1,2\alpha}$ and

$$d\mathcal{E}_H^\alpha(u)h = \int_D (1 + |\nabla u|^2)^{\alpha-1} \nabla u \cdot \nabla h + 2 \int_D H(u) h \cdot u_x \wedge u_y$$

for every $u, h \in H_0^{1,2\alpha}$ (see [11]).

Our first goal is to prove that for every $\alpha > 1$ sufficiently close to 1 the functional $\mathcal{E}_H^\alpha$ has a mountain pass geometry and a corresponding mountain pass critical point, as stated in the following result.

Lemma 3.1 *Let $H \in C^1(\mathbb{R}^3) \cap L^\infty$ be such that there exists $\bar{u} \in C^\infty_c(D, \mathbb{R}^3)$ with $\mathcal{E}_H(\bar{u}) < 0$. Then there exists $\bar{\alpha} > 1$ such that for every $\alpha \in (1, \bar{\alpha})$ the class $\Gamma^\alpha = \{\gamma \in C([0,1], H_0^{1,2\alpha}) : \gamma(0) = 0,\ \mathcal{E}^\alpha_H(\gamma(1)) < 0\}$ is nonempty and the value*

$$\overline{c}^\alpha_H = \inf_{\gamma \in \Gamma^\alpha} \max_{s \in [0,1]} \mathcal{E}^\alpha_H(\gamma(s))$$

is positive.
If in addition $\bar{M}_H < +\infty$ then for every $\alpha \in (1, \bar{\alpha})$ there exists $u^\alpha \in H_0^{1,2\alpha}$ such that $\mathcal{E}^\alpha_H(u^\alpha) = \overline{c}^\alpha_H$ and $d\mathcal{E}^\alpha_H(u^\alpha) = 0$.

The second step consists in obtaining some uniform estimates on the mountain pass critical points u^α of the perturbed functionals $\mathcal{E}^\alpha_H$.

Proposition 3.2 *Let $H \in C^1(\mathbb{R}^3)$ be such that $\bar{M}_H < 1$ and, for every $\alpha \in (1, \bar{\alpha})$, let $u^\alpha \in H^{1,2\alpha}$ be the critical point of $\mathcal{E}^\alpha_H$ at level $\overline{c}^\alpha_H$ given by Lemma 3.1. Then*

$$\begin{aligned} &\limsup_{\alpha \to 1} \mathcal{E}^\alpha_H(u^\alpha) \le c_H \ , \\ &\sup_{\alpha \in (1,\bar{\alpha})} \|\nabla u^\alpha\|_2 < +\infty \ , \\ &\inf_{\alpha \in (1,\bar{\alpha})} \|\nabla u^\alpha\|_2 > 0 \ , \end{aligned}$$

where c_H is defined by (2.6). If, in addition, $H(u) = H_0$ for $|u| \ge R_0$, for some $R_0 > 0$, then

$$\sup_{\alpha \in (1,\bar{\alpha})} \|u^\alpha\|_\infty < +\infty \ .$$

The proofs of Lemma 3.1 and Proposition 3.2 will be carried out in Subsections 3.1 and 3.2, respectively.

The last result of this Section states the behaviour of the family of the mountain pass critical points u^α in the limit as $\alpha \to 1$. This result describes a blow up phenomenon, and it will be proved in the Appendix, in a more general situation.

Proposition 3.3 *Let $H \in C^1(\mathbb{R}^3)$ be such that $\bar{M}_H < 1$ and $H(u) = H_0$ for $|u| \ge R_0$, for some $R_0 > 0$. For every $\alpha \in (1, \bar{\alpha})$, let $u^\alpha \in H^{1,2\alpha}$ be the critical point of $\mathcal{E}^\alpha_H$ at level $\overline{c}^\alpha_H$ given by Lemma 3.1. Then, there exist sequences $(\epsilon_\alpha) \subset (0, +\infty)$, $(z_\alpha) \subset \overline{D}$, a number $\lambda \in (0, 1]$, and a function $\omega \in X \cap L^\infty$ such that, setting $v^\alpha(z) = u^\alpha(\epsilon_\alpha z + z_\alpha)$, for a subsequence, one has:*

(i) $\epsilon_\alpha \to 0$ *and* $\epsilon_\alpha^{2(\alpha-1)} \to \lambda$,

(ii) $v^\alpha \to \omega$ *strongly in* $H^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$ *and uniformly on compact sets of* $\mathbb{R}^2$,

(iii) ω *is a nonconstant solution to* $\Delta\omega = 2\lambda H(\omega)\omega_x \wedge \omega_y$ *on* $\mathbb{R}^2$,

(iv) $\mathcal{E}_{\lambda H}(\omega) \le \lambda \liminf_{\alpha \to 1} \mathcal{E}^\alpha_H(u^\alpha)$.

3.1. *Proof of Lemma 3.1*

Lemma 3.4 *Let $\rho \in (0, (\frac{S_H}{2})^{3/2}]$ being S_H given by (2.3). Then, for every $u \in H_0^{1,2\alpha}$ such that $\|\nabla u\|_2 \le \rho$ one has $\mathcal{E}_H^\alpha(u) \ge \frac{1}{2}\mathcal{D}(u)$.*

Proof. Using the inequality $(1+s^2)^\alpha \ge 1+\alpha s^2$, one infers that $\mathcal{E}_H^\alpha(u) \ge \mathcal{E}_H(u)$ for every $u \in H_0^{1,2\alpha}$. In addition, by the isoperimetric inequality (2.3) one has $\mathcal{E}_H(u) \ge \mathcal{D}(u) - 2S_H^{-3/2}\mathcal{D}(u)^{3/2}$. Therefore $\|\nabla u\|_2 \le \rho$ implies $\mathcal{E}_H^\alpha(u) \ge (1-\sqrt{2}S_H^{-3/2}\rho)\mathcal{D}(u)$ and the thesis follows since $\rho \le (\frac{S_H}{2})^{3/2}$. □

Lemma 3.5 *If $u \in H_0^{1,2\bar\alpha}$ for some $\bar\alpha > 1$, then $\mathcal{E}_H^\alpha(su) \to \mathcal{E}_H(su)$ as $\alpha \to 1$, uniformly with respect to $s \in [0,\bar s]$ for every $\bar s > 0$.*

Proof. The thesis follows by the estimate

$$\begin{aligned} 0 \;&\le\; \mathcal{E}_H^\alpha(su) - \mathcal{E}_H(su) = \frac{1}{2\alpha}\int_D \left((1+s^2|\nabla u|^2)^\alpha - 1 - \alpha s^2|\nabla u|^2\right) \\ &\le\; \frac{1}{2\alpha}\int_D \left(2^{\alpha-1} - 1 + 2^{\alpha-1}s^{2\alpha}|\nabla u|^{2\alpha} - \alpha s^2|\nabla u|^2\right) \;, \end{aligned}$$

and by standard techniques. □

Lemma 3.6 *If $\bar M_H < +\infty$ then for $\alpha \in (1, \frac{3}{2})$ the functional $\mathcal{E}_H^\alpha$ satisfies the Palais-Smale condition on $H_0^{1,2\alpha}$.*

Proof. First, note that for every $u \in H_0^{1,2\alpha}$, using (2.8) one has

$$\begin{aligned} 3\mathcal{E}_H^\alpha(u) - d\mathcal{E}_H^\alpha(u)u \;&\ge\; \left(\frac{3}{2\alpha} - 1\right)\mathcal{D}^\alpha(u) + 2\int_D (3m_H(u) - H(u))u \cdot u_x \wedge u_y \\ &\ge\; \left(\frac{3}{2\alpha} - 1\right)\|\nabla u\|_{2\alpha}^{2\alpha} - \frac{\bar M_H}{2}\|\nabla u\|_2^2 \;. \end{aligned}$$

Hence,

$$\left(\frac{3}{2\alpha} - 1\right)\|\nabla u\|_{2\alpha}^{2\alpha} \le \bar M_H C_\alpha \|\nabla u\|_{2\alpha}^2 + \|d\mathcal{E}_H^\alpha(u)\|\;\|\nabla u\|_{2\alpha} + 3\mathcal{E}_H^\alpha(u) \;. \tag{3.1}$$

Now, let $(u^n) \subset H_0^{1,2\alpha}$ be a Palais-Smale sequence for $\mathcal{E}_H^\alpha$. By (3.1) the sequence (u^n) is bounded in $H_0^{1,2\alpha}$. Then, there exists $\bar u \in H_0^{1,2\alpha}$ such that (for a subsequence) $u^n \to \bar u$ weakly in $H^{1,2\alpha}$ and uniformly on $\overline{D}$ (by Rellich Theorem). We need the following auxiliary result (see [3], for a proof):

Lemma 3.7 *Let $(u^n), (v^n) \subset H_0^1 \cap L^\infty$ be such that $u^n \to u$ weakly in H^1 and $v^n \to v$ uniformly. Then*

$$\int_D v^n \cdot u_x^n \wedge u_y^n \to \int_D v \cdot u_x \wedge u_y \;.$$

Since for every $h \in H_0^{1,2\alpha}$

$$\int_D (1+|\nabla u^n|^2)^{\alpha-1} \nabla u^n \cdot \nabla h + 2\int_D H(u^n) h \cdot u_x^n \wedge u_y^n \to 0$$

as $n \to +\infty$, thanks to Lemma 3.7 we obtain that $d\mathcal{E}_H^\alpha(\bar{u}) = 0$. In particular $0 = d\mathcal{E}_H^\alpha(\bar{u})(u^n - \bar{u}) = d\mathcal{D}^\alpha(u^n)(u^n - \bar{u}) + o(1)$. On the other hand, we can use again Lemma 3.7 to get $o(1) = d\mathcal{E}_H^\alpha(u^n)(u^n - \bar{u}) = d\mathcal{D}^\alpha(u^n)(u^n - \bar{u}) + o(1)$. Therefore, $(d\mathcal{D}^\alpha(u^n) - d\mathcal{D}^\alpha(\bar{u}))(u^n - \bar{u}) = o(1)$. Finally we note that $\mathcal{D}^\alpha$ is strictly convex on $H_0^{1,2\alpha}$, and hence $d\mathcal{D}^\alpha$ is strictly monotone. This readily leads to the conclusion. □

In conclusion, we notice that the first part of Lemma 3.1 is an immediate consequence of Lemmata 3.4 and 3.5. The existence of the critical point u^α is obtained as an application of the mountain pass theorem, and by Lemma 3.6.

3.2. *Proof of Proposition 3.2*

In order to show the first estimate, it is useful to introduce, for every $\alpha \in (1, \bar{\alpha})$, the value

$$c_H^\alpha = \inf_{\substack{u \in H_0^{1,2\alpha} \\ u \neq 0}} \sup_{s>0} \mathcal{E}_H^\alpha(su) \ .$$

Lemma 3.8 *Let $H \in C^1(\mathbb{R}^3)$ satisfy $\bar{M}_H < 1$. Then* $\limsup_{\alpha \to 1} c_H^\alpha \le c_H$.

Proof. Fix $\epsilon > 0$ and take $u \in C_c^\infty(D, \mathbb{R}^3)$ such that $\sup_{s>0} \mathcal{E}_H(su) < c_H + \epsilon$. For every $s \ge 0$, using Lemma 2.9, one has

$$\begin{aligned} \mathcal{E}_H^\alpha(su) &= \mathcal{D}^\alpha(su) - \mathcal{D}(su) + \mathcal{E}_H(su) \\ &\le C_0(s^{2\alpha} + 1) - C_1 s^3 \end{aligned} \tag{3.2}$$

with $C_0, C_1 > 0$ depending just on u (and not on α). Therefore, for $\alpha \in (1, \frac{3}{2})$ there exists $\bar{s}_\alpha > 0$ such that $\mathcal{E}_H^\alpha(\bar{s}_\alpha u) = \sup_{s>0} \mathcal{E}_H^\alpha(su)$. From (3.2) it follows that $\bar{s}_\alpha$ is uniformly bounded. Then, by Lemma 3.5, $\lim_{\alpha \to 1} \mathcal{E}_H^\alpha(\bar{s}_\alpha u) = \mathcal{E}_H(\bar{s}u)$ for some $\bar{s} > 0$. Hence, $\limsup_{\alpha \to 1} c_H^\alpha \le c_H + \epsilon$ and the thesis follows by the arbitrariness of $\epsilon > 0$. □

Concerning the H_0^1 bounds we have the following result.

Lemma 3.9 *Let $H \in C^1(\mathbb{R}^3)$ satisfy $\bar{M}_H < 1$. If $u \in H_0^{1,2\alpha}$ is a nonzero critical point of $\mathcal{E}_H^\alpha$, then*

$$\frac{1}{2}\left(\frac{2 - \bar{M}_H}{3}\right)^2 S_H^3 \le \int_D |\nabla u|^2 \le \left(\frac{1}{2\alpha} - \frac{1}{3} - \frac{\bar{M}_H}{6}\right)^{-1} \mathcal{E}_H^\alpha(u) \ ,$$

where S_H is given by (2.3).

Proof. Using (2.8) one has

$$3\mathcal{E}_H^\alpha(u) \ = \ 3\mathcal{D}^\alpha(u) - d\mathcal{D}^\alpha(u)u + 2\int_D (3m_H(u) - H(u))u \cdot u_x \wedge u_y$$

$$\begin{aligned}
&\geq \left(\frac{3}{2\alpha}-1\right)\int_D \left(1+|\nabla u|^2\right)^{\alpha-1}|\nabla u|^2 + \frac{3}{2\alpha}\int_D \left(\left(1+|\nabla u|^2\right)^{\alpha-1}-1\right) \\
&\quad -\frac{\bar{M}_H}{2}\int_D |\nabla u|^2 \\
&\geq \left(\frac{3}{2\alpha}-1-\frac{\bar{M}_H}{2}\right)\int_D |\nabla u|^2 \ .
\end{aligned}$$

Moreover, by (2.3) and (2.8) again, one has

$$\begin{aligned}
2\mathcal{D}(u) &\leq \int_D \left(1+|\nabla u|^2\right)^{\alpha-1}|\nabla u|^2 \\
&= -6\mathcal{V}_H(u) + 2\int_D (3m_H(u)-H(u))\, u\cdot u_x\wedge u_y \\
&\leq 6S_H^{-\frac{3}{2}}\mathcal{D}(u)^{\frac{3}{2}} + \bar{M}_H\mathcal{D}(u) \ .
\end{aligned}$$

Since $u \neq 0$ one gets the thesis. □

Finally, to show the L^∞ bound, H is asked to be constant far out and the following estimate holds.

Lemma 3.10 *Let $H \in C^1(\mathbb{R}^3)$ be such that $H(u) = H_0$ for $|u| \geq R_0$, where $R_0 > 0$ is given. If $u \in H_0^{1,2\alpha}$ is a critical point of $\mathcal{E}_H^\alpha$, then*

$$\|u\|_\infty \leq C|H_0| \ \|\nabla u\|_2^2 + R_0$$

where C is a universal positive constant (independent of α, R_0, H_0 and u).

Proof. If $u \in H_0^{1,2\alpha}$ is a critical point of $\mathcal{E}_H^\alpha$, then u is a weak solution to problem

$$\begin{cases} \operatorname{div}(a_\alpha(z)\nabla u) = 2H(u)u_x\wedge u_y & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

where $a_\alpha(z) = (1+|\nabla u(z)|^2)^{\alpha-1}$. Fix $R > R_0$ and let Ω_0 be a component of $\{z \in D : |u(z)| > R\}$, if there exists. Since u is continuous, the set Ω_0 is nonempty, bounded, open and connected, and $|u| = R$ on $\partial\Omega_0$. Taking $\delta \in (0, R-R_0)$ one can find a bounded, smooth domain $\Omega = \Omega_\delta$ close to Ω_0 such that $|u(z)| > R_0$ for $z \in \Omega$ and $|u(z)| \leq R+\delta$ for $z \in \partial\Omega$. Hence u satisfies

$$\operatorname{div}(a_\alpha(z)\nabla u) = 2H_0 u_x\wedge u_y \quad \text{on } \Omega \ . \tag{3.3}$$

For every $k \in \mathbb{N}$ let $a_\alpha^k = \min\{a_\alpha, k\}$ and let φ^k be the solution to problem

$$\begin{cases} \operatorname{div}(a_\alpha^k(z)\nabla\varphi) = g & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

where $g = 2H_0u_x\wedge u_y$. Since a_α^k is a continuous bounded function on Ω and $a_\alpha^k \geq 1$, by a result of Bethuel and Ghidaglia, Theorem 1.3 in [1], there exists a constant $C > 0$ such that

$$\|\varphi^k\|_\infty + \|\nabla\varphi^k\|_2 \leq C|H_0| \ \|\nabla u\|_2^2 \tag{3.5}$$

and C is independent of k, α, Ω and u. Hence the sequence (φ^k) is bounded in $H^1_0(\Omega)$ and thus, there exists $\varphi \in H^1_0(\Omega)$ such that, for a subsequence, $\varphi^k \to \varphi$ weakly in $H^1_0(\Omega)$ and pointwise a.e. We remark that $a_\alpha \in L^{\frac{\alpha}{\alpha-1}}(\Omega)$ since $u \in H_0^{1,2\alpha}$. In particular $a_\alpha \in L^2(\Omega)$ for $\alpha < 2$ and $a^k_\alpha \to a_\alpha$ strongly in $L^2(\Omega)$. By (3.4) for every $h \in C^\infty_c(\Omega)$

$$\int_\Omega a^k_\alpha(z)\nabla\varphi^k \cdot \nabla h = -\int_\Omega gh \ .$$

Hence, by a standard limit procedure, we obtain that for every $h \in C^\infty_c(\Omega)$

$$\int_\Omega a_\alpha(z)\nabla\varphi \cdot \nabla h = -\int_\Omega gh$$

that is, φ is a weak solution to

$$\begin{cases} \operatorname{div}(a_\alpha(z)\nabla\varphi) = g & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \ . \end{cases} \tag{3.6}$$

Moreover, by (3.5) we also get

$$\|\varphi\|_\infty + \|\nabla\varphi\|_2 \le C|H_0| \ \|\nabla u\|_2^2 \ . \tag{3.7}$$

Now, we observe that, thanks to (3.3) and (3.6), the function $\psi = u - \varphi$ is the solution to problem

$$\begin{cases} \operatorname{div}(a_\alpha(z)\nabla\psi) = 0 & \text{in } \Omega \\ \psi = u & \text{on } \partial\Omega \end{cases} \tag{3.8}$$

and it can be characterized as the minimum for the problem

$$\inf\Big\{ \int_\Omega a_\alpha(z)|\nabla\psi|^2 : \psi \in u + H^1_0(\Omega)\Big\} \ . \tag{3.9}$$

Hence $\|\psi\|_\infty \le R + \delta$. Otherwise, if P denotes the projection on the disc $D_{R+\delta}$, that is

$$P(z) = \begin{cases} z & \text{if } |z| \le R+\delta \\ (R+\delta)\frac{z}{|z|} & \text{if } |z| > R+\delta, \end{cases}$$

then $\bar\psi = P \circ \psi$ will be a solution to (3.9) and then to (3.8). In conclusion, using (3.7),

$$\|u\|_\infty \le \|\varphi\|_\infty + \|\psi\|_\infty \le C|H_0| \ \|\nabla u\|_2^2 + R + \delta \ ,$$

and by the arbitrariness of $R > R_0$ and $\delta > 0$ one gets the thesis. □

Finally, Proposition 3.2 follows by Lemmata 3.8, 3.9 and 3.10, noting that $c^\alpha_H \ge \overline{c}^\alpha_H = \mathcal{E}^\alpha_H(u^\alpha)$.

4. Proof of the main theorem

Here we give the proof of Theorem 1.1. First, as a preliminary result we consider the case in which H is constant outside a ball (Subsection 4.1). Then, in Subsection 4.2, we remove this condition, just asking H to be asymptotic to a constant at infinity, according to the assumption $(\mathbf{h}_\infty)$.

4.1. *Case H constant far out*

The results obtained in the previous Sections allow us to deduce the existence of an H-bubble when the prescribed curvature H satisfies $(\mathbf{h_1})$ and is constant far out (this last condition enters in order to guarantee an L^∞ bound on the approximating solutions). More precisely, the following result holds.

Theorem 4.1 *Let $H \in C^1(\mathbb{R}^3)$ verify $(\mathbf{h_1})$ and the following conditions:*

(i) there exists $\bar{u} \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ such that $\mathcal{E}_H(\bar{u}) < 0$,

(ii) there exists $R_0 > 0$ and $H_0 \in \mathbb{R}$ such that $H(u) = H_0$ as $|u| \geq R_0$.

Then there exists an H-bubble ω such that $\mathcal{E}_H(\omega) = c_H$, where c_H is defined by (2.6).

Remark 4.2 Suppose that in the assumption (ii) $H_0 \neq 0$. Then, by Lemma 2.5, the condition (i) is automatically fulfilled. Moreover, in this case problem (1.1) admits the (trivial) solution ω^0 which parametrizes a sphere of radius $|H_0|^{-1}$ placed in the region $|u| > R_0$. However, the additional information on the energy of the H-bubble ω makes meaningful the above result, since if $c_H < \frac{4\pi}{3H_0^2}$ then ω is geometrically different from ω^0.

Proof. From the assumptions (i) and (ii), and since $\bar{M}_H < 1$, thanks to Propositions 3.2 and 3.3, there exists a function $\omega \in X \cap L^\infty$ which is a λH-bubble with $\lambda \in (0, 1]$ and $\mathcal{E}_{\lambda H}(\omega) \leq \lambda c_H$. Since $M_H < 1$, by Proposition 2.8 (applied with λH instead of H), $\mathcal{E}_{\lambda H}(\omega) \geq c_{\lambda H}$. Finally, Lemma 2.10 implies $\mathcal{E}_{\lambda H}(\omega) \geq c_H$. Then $\lambda = 1$ and the Theorem is proved. □

4.2. *General case*

Now we want to remove the hypothesis that H is constant far out, by requiring just an asymptotic behaviour at infinity as stated by $(\mathbf{h}_\infty)$. To this aim, we will use the condition $(*)$. Our argument consists in approximating H with a sequence of functions $(H_n) \subset C^1(\mathbb{R}^3)$ satisfying the hypotheses of Theorem 4.1 and then passing to the limit on the sequence (ω^n) of the corresponding H_n-bubbles. The information on the energies $\mathcal{E}_{H_n}(\omega^n)$ together with the condition $(*)$ will permit us to obtain some L^∞ bound on the sequence (ω^n), and then to get the result.

Thus, let us start with the construction of the sequence (H_n).

Lemma 4.3 *Let $H \in C^1(\mathbb{R}^3)$ satisfying $(\mathbf{h}_\infty)$ and let M_H be defined by* (2.7). *Then there exists a sequence $(H_n) \subset C^1(\mathbb{R}^3)$ such that:*

(i) $H_n \to H$ uniformly on $\mathbb{R}^3$,

(ii) for every $n \in \mathbb{N}$ there exists $R_n > 0$ such that $H_n(u) = H_\infty$ as $|u| \geq R_n$,

(iii) $\sup_{u \in \mathbb{R}^3} |\nabla H_n(u) \cdot u\, u| := M_{H_n} \leq M_H$.

Proof. It is not restrictive to suppose $H_\infty = 0$. Hence, for every $u \in \mathbb{R}^3 \setminus \{0\}$ one has

$$H(u) = -\int_1^{+\infty} \nabla H(su) \cdot u \; ds \; .$$

Let $\chi \in C^\infty(\mathbb{R}, [0,1])$ be such that $\chi(r) = 1$ as $r \leq 0$, $\chi(r) = 0$ as $r \geq 1$ and $|\chi'| \leq 2$. Given any sequence $r_n \to +\infty$ set $\chi_n(r) = \chi(r - r_n)$ and

$$H_n(u) = -\int_1^{+\infty} \chi_n(s|u|) \nabla H(su) \cdot u \; ds$$

for every $u \in \mathbb{R}^3 \setminus \{0\}$. By continuity, H_n is well defined and continuous on $\mathbb{R}^3$. In fact $H_n \in C^1(\mathbb{R}^3)$ and for each $u \in \mathbb{R}^3 \setminus \{0\}$

$$\nabla H_n(u) \cdot u = \frac{d}{ds} H_n(su)\Big|_{s=1} = \chi_n(|u|) \nabla H(u) \cdot u \; . \tag{4.1}$$

Therefore (iii) holds true. By the definition of H_n, one has $H_n(u) = 0$ as $|u| > r_n + 1$. Thus (ii) follows, with $R_n = r_n + 1$. Moreover (4.1) implies (iii). Now, notice that

$$H_n(u) = \chi_n(|u|) H(u) + \int_{r_n}^{r_n+1} \chi_n'(t) H\left(t \frac{u}{|u|}\right) \; dt \; . \tag{4.2}$$

Setting $\epsilon_n = \sup_{|u| \geq r_n} |H(u)|$, one has that

$$\begin{aligned} \left| \int_{r_n}^{r_n+1} \chi_n'(t) H\left(t \frac{u}{|u|}\right) \; dt \right| &\leq 2\epsilon_n \\ |(\chi_n(|u|) - 1) H(u)| &\leq 2\epsilon_n \; . \end{aligned}$$

Hence, (4.2) implies that $|H_n(u) - H(u)| \leq 4\epsilon_n$ for every $u \in \mathbb{R}^3$ and then, since $\epsilon_n \to 0$, also (i) is proved. □

As a further tool, we also need the following result.

Lemma 4.4 *Let* $(\tilde{H}_n) \subset C^1(\mathbb{R}^3)$, $H_\infty \in \mathbb{R}$ *and* $(\tilde{\omega}_n) \subset X \cap L^\infty$ *be such that:*

(i) $\tilde{H}_n \to H_\infty$ *uniformly on compact sets,*

(ii) $\sup_n (\|\nabla \tilde{\omega}^n\|_2 + \|\tilde{\omega}^n\|_\infty) < +\infty$,

(iii) *for every* $n \in \mathbb{N}$ *the function* $\tilde{\omega}^n$ *solves* $\Delta \tilde{\omega}^n = 2\tilde{H}_n(\tilde{\omega}^n) \tilde{\omega}^n_x \wedge \tilde{\omega}^n_y$ *on* $\mathbb{R}^2$.

Then $H_\infty \neq 0$ *and* $\liminf \mathcal{E}_{\tilde{H}_n}(\tilde{\omega}^n) \geq \frac{4\pi}{3H_\infty^2}$.

Proof. From the assumption (ii), there exists $\omega \in X \cap L^\infty$ such that, for a subsequence, $\nabla \omega^n \to \nabla \omega$ weakly in $(L^2(\mathbb{R}^2, \mathbb{R}^3))^2$. Thanks to the invariance of H-systems with respect to dilations, translations and Kelvin transform, we may also assume that $\|\nabla \tilde{\omega}^n\|_\infty = |\nabla \tilde{\omega}^n(0)| = 1$. Then, arguing as in the proof of Proposition A.1, using the hypotheses (i)–(iii), one can show that ω is an H_∞-bubble, and $\tilde{\omega}^n \to \omega$

strongly in $H^1_{loc}(\mathbb{R}^2,\mathbb{R}^3)$ and in $L^\infty_{loc}(\mathbb{R}^2,\mathbb{R}^3)$. In particular it must be $H_\infty \neq 0$ (there exists no 0-bubble in X). Furthermore, for every $R>0$, one has

$$\mathcal{E}_{\tilde{H}_n}(\tilde{\omega}^n, D_R) \quad\rightarrow\quad \mathcal{E}_{H_\infty}(\omega, D_R) \tag{4.3}$$

$$\int_{\partial D_R} \tilde{\omega}^n \cdot \frac{\partial \tilde{\omega}^n}{\partial \nu} \quad\rightarrow\quad \int_{\partial D_R} \omega \cdot \frac{\partial \omega}{\partial \nu}\ , \tag{4.4}$$

where, in (4.3), we used the notation:

$$\mathcal{E}_H(u,\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2 + 2\int_\Omega m_H(u) u \cdot u_x \wedge u_y\ .$$

Now, fixing $\epsilon > 0$, let $R>0$ be such that

$$|\mathcal{E}_{H_\infty}(\omega, \mathbb{R}^2\setminus D_R)| < \epsilon\ , \quad (|H_\infty|\ \|\omega\|_\infty + 1)\int_{\mathbb{R}^2\setminus D_R} |\nabla\omega|^2 < \epsilon\ .$$

Multiplying $\Delta\omega = 2H_\infty \omega_x \wedge \omega_y$ by ω and integrating over $\mathbb{R}^2\setminus D_R$ we find

$$\left|\int_{\partial D_R} \omega\cdot\frac{\partial\omega}{\partial\nu}\right| = \left|\int_{\mathbb{R}^2\setminus D_R}\left(\omega\cdot\Delta\omega + |\nabla\omega|^2\right)\right| \le \epsilon\ .$$

Then, by (4.4), one has that

$$\left|\int_{\partial D_R} \tilde{\omega}^n \cdot \frac{\partial \tilde{\omega}^n}{\partial \nu}\right| \le \epsilon + o(1)\ . \tag{4.5}$$

Now we multiply $\Delta\tilde{\omega}^n = 2\tilde{H}_n(\tilde{\omega}^n)\tilde{\omega}^n_x \wedge \tilde{\omega}^n_y$ by $\tilde{\omega}^n$ and we integrate over $\mathbb{R}^2\setminus D_R$ to get

$$\begin{aligned}
\int_{\partial D_R} \tilde{\omega}^n \cdot \frac{\partial \tilde{\omega}^n}{\partial \nu} &= \int_{\mathbb{R}^2\setminus D_R} |\nabla\tilde{\omega}^n|^2 + 2\int_{\mathbb{R}^2\setminus D_R} \tilde{H}_n(\tilde{\omega}^n)\tilde{\omega}^n\cdot\tilde{\omega}^n_x\wedge\tilde{\omega}^n_y \\
&= 3\mathcal{E}_{\tilde{H}_n}(\tilde{\omega}^n, \mathbb{R}^2\setminus D_R) - \frac{1}{2}\int_{\mathbb{R}^2\setminus D_R}|\nabla\tilde{\omega}^n|^2 \\
&\quad +2\int_{\mathbb{R}^2\setminus D_R}(\tilde{H}_n(\tilde{\omega}^n) - 3m_{\tilde{H}_n}(\tilde{\omega}^n))\tilde{\omega}^n\cdot\tilde{\omega}^n_x\wedge\tilde{\omega}^n_y \\
&\le 3\mathcal{E}_{\tilde{H}_n}(\tilde{\omega}^n, \mathbb{R}^2\setminus D_R) - \left(\frac{1}{2} - \mu_n\rho\right)\int_{\mathbb{R}^2\setminus D_R}|\nabla\tilde{\omega}^n|^2
\end{aligned} \tag{4.6}$$

where $\rho = \sup_n \|\tilde{\omega}^n\|_\infty$, and $\mu_n = \sup_{|u|\le\rho}|\tilde{H}_n(u) - 3m_{\tilde{H}_n}(u)|$. Hence (4.5) and (4.6) imply

$$\mathcal{E}_{\tilde{H}_n}(\tilde{\omega}^n, \mathbb{R}^2\setminus D_R) \ge -\frac{\epsilon}{3} + o(1)\ ,$$

because, by (i), $\mu_n \to 0$. Finally, we have

$$\begin{aligned}
\mathcal{E}_{H_\infty}(\omega) - \epsilon &\le \mathcal{E}_{H_\infty}(\omega, D_R) = \mathcal{E}_{\tilde{H}_n}(\tilde{\omega}^n, D_R) + o(1) \\
&= \mathcal{E}_{\tilde{H}_n}(\tilde{\omega}^n) - \mathcal{E}_{\tilde{H}_n}(\tilde{\omega}^n, \mathbb{R}^2\setminus D_R) + o(1) \le \mathcal{E}_{\tilde{H}_n}(\tilde{\omega}^n) + \frac{\epsilon}{3} + o(1)\ .
\end{aligned}$$

Hence, by the arbitrariness of $\epsilon > 0$, one obtains $\liminf \mathcal{E}_{\tilde{H}_n}(\tilde{\omega}^n) \geq \mathcal{E}_{H_\infty}(\omega)$ and the thesis follows by Remark 2.6. □

Proof of Theorem 1.1. Let $(H_n) \subset C^1(\mathbb{R}^3)$ be the sequence given by Lemma 4.3. From Theorem 4.1, for every n there exists an H_n-bubble ω^n such that $\mathcal{E}_{H_n}(\omega^n) = c_{H_n}$. By Lemma 2.11, one has that

$$\limsup_{n\to+\infty} \mathcal{E}_{H_n}(\omega^n) \leq c_H \ . \tag{4.7}$$

We point out that if we prove that $\sup_n \|\omega^n\|_\infty = R < +\infty$, then we have concluded, since for n large, $H(u) = H_n(u)$ as $|u| \leq R$. To this goal, as a first step, we show that

$$\|\omega^n - \omega^n_\infty\|_\infty \leq C_1 \left(1 + \int_{\mathbb{R}^2} |\nabla\omega^n|^2\right) \tag{4.8}$$

where $\omega^n_\infty = \lim_{|z|\to\infty} \omega^n(z)$ and $C_1 > 0$ depends only on $\|H\|_\infty$. This is a consequence of an *a priori* L^∞ estimate proved by Grüter [7] (see also Theorem 4.8 in [2]). More precisely, fixing an arbitrary $\delta > 0$, for every n there exists $\rho_n > 0$, depending on δ, such that if $|z| \geq \rho_n$ then $|\omega^n(z) - \omega^n_\infty| \leq \delta$. Let us set

$$\begin{aligned} \gamma^n(z) &= \omega^n(\rho_n z) - \omega^n_\infty \quad \text{as } z \in \partial D \\ u^n(z) &= \omega^n(\rho_n z) - \omega^n_\infty \quad \text{as } z \in D \ . \end{aligned}$$

Thus u^n is a smooth and conformal solution to

$$\begin{cases} \Delta u^n = 2\tilde{H}_n(u^n) u^n_x \wedge u^n_y & \text{in } D \\ u^n = \gamma^n & \text{on } \partial D \ , \end{cases}$$

where $\tilde{H}_n(u) = H_n(u + \omega^n_\infty)$. Hence, by [7],

$$\|u^n\|_{L^\infty(D)} \leq \|\gamma^n\|_{L^\infty(\partial D)} + C\left(1 + \int_D |\nabla\omega^n|^2\right)$$

with $C > 0$ depending on $\|\tilde{H}_n\|_\infty = \|H_n\|_\infty$. Since $H_n \to H$ uniformly on $\mathbb{R}^3$, actually, C is independent of n, but depends only on $\|H\|_\infty$. Then

$$\|\omega^n - \omega^n_\infty\|_{L^\infty(\mathbb{R}^2)} \leq \delta + \|u^n\|_{L^\infty(D)} \leq 2\delta + C\left(1 + \int_{\mathbb{R}^2} |\nabla\omega^n|^2\right) \ .$$

Therefore (4.8) holds true. As a second step, we show that for every n

$$\int_{\mathbb{R}^2} |\nabla\omega^n|^2 \leq C_2 \tag{4.9}$$

where $C_2 > 0$ depends only on H. Indeed, by (2.7), using (2.4), one has $(1 - M_{H_n})\mathcal{D}(\omega^n) \leq 3\mathcal{E}_{H_n}(\omega^n)$. Since $M_{H_n} \leq M_H$, from (4.7) it follows that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^2} |\nabla\omega^n|^2 \leq \frac{6c_H}{1 - M_H} \ ,$$

and thus (4.9) is proved. Consequently, by (4.8), one obtains

$$\|\omega^n - \omega^n_\infty\|_\infty \le C_3 \tag{4.10}$$

with $C_3 > 0$ independent of n. As a last step, let us show that $\sup_n |\omega^n_\infty| < +\infty$. We argue by contradiction, assuming that (for a subsequence) $|\omega^n_\infty| \to +\infty$. Since $H_n \to H$ uniformly on $\mathbb{R}^3$, by $(\mathbf{h}_\infty)$, we have that $\tilde{H}_n \to H_\infty$ uniformly on compact sets. Moreover, $\tilde{\omega}^n(z) = \omega^n(z) - \omega^n_\infty$ is an $\tilde{H}_n$-bubble and, thanks to (4.10) and (4.9), we can apply Lemma 4.4, to infer that $H_\infty \ne 0$ and

$$\frac{4\pi}{3H^2_\infty} \le \liminf \mathcal{E}_{\tilde{H}_n}(\tilde{\omega}^n) = \liminf \mathcal{E}_{H_n}(\omega^n)\ .$$

Then (4.7) implies that $\frac{4\pi}{3H^2_\infty} \le c_H$, contrary to the condition $(*)$. Therefore, we have that $\sup |\omega^n_\infty| < +\infty$, that, together with (4.10), gives the desired estimate. This concludes the proof. □

We end the work, by making some comments about the case of *radially symmetric* curvatures.

Example 4.5 Let $H \in C^1(\mathbb{R}^3)$ be a radial function satisfying $(\mathbf{h_1})$ and $(\mathbf{h}_\infty)$ with $H_\infty \ne 0$. Given $\phi\colon \mathbb{R}^2 \to \mathbb{S}^2$ defined by (2.1), and $\rho > 0$, the mapping $\rho\phi$ is a solution to (1.1), i.e., it is a radial H-bubble, if and only if $\rho|H(\rho)| = 1$. In this case the energy of this radial H-bubble is $\frac{4\pi}{3H(\rho)^2}$. Clearly, since H is regular and $H_\infty \ne 0$, the equation $\rho|H(\rho)| = 1$ always admits positive solutions. Now, suppose, in addition that the condition $(*)$ holds true. This happens, for instance, if $H(\rho) > H_\infty > 0$ for ρ large. Then, there exist H-bubbles with minimal energy $c_H < \frac{4\pi}{3H^2_\infty}$. Hence, these minimal H-bubbles cannot be radial if $|H(\rho)| \le H_\infty$ whenever $\rho|H(\rho)| = 1$.

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A. Appendix

A.1. *Convergence of approximating solutions in a Sacks-Uhlenbeck type setting*

In this Appendix we study the behaviour of sequences of solutions of approximating problems of the type

$$\begin{cases} \operatorname{div}((1+|\nabla u|^2)^{\alpha-1}\nabla u) = 2H(u)u_x \wedge u_y & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

in the limit as $\alpha \to 1_+$. More precisely, we assume that for every $\alpha \in (1, \bar{\alpha})$ a function $u^\alpha \in H^{1,2\alpha}_0$ is given, in such a way that

$$d\mathcal{E}^\alpha_H(u^\alpha) = 0, \tag{A.1}$$

$$\sup_{\alpha\in(1,\bar\alpha)}(\|u^\alpha\|_\infty+\|\nabla u^\alpha\|_2)<+\infty\ , \tag{A.2}$$

$$\inf_{\alpha\in(1,\bar\alpha)}\|\nabla u^\alpha\|_2>0\ . \tag{A.3}$$

The first main result is non-variational and concerns a blow up analysis of sequences of approximating solutions. We point out that this result applies to *any* sequence of functions satisfying (A.1)–(A.3).

Proposition A.1 *Let $H\in C^1(\mathbb{R}^3)\cap L^\infty$ and for every $\alpha\in(1,\bar\alpha)$ let $u^\alpha\in H_0^{1,2\alpha}$ satisfy* (A.1)–(A.3)*. Then, there exist sequences $(\epsilon_\alpha)\subset(0,+\infty)$, $(z_\alpha)\subset\overline{D}$, a number $\lambda\in(0,1]$, and a function $\omega\in X\cap L^\infty$ such that, setting $v^\alpha(z)=u^\alpha(\epsilon_\alpha z+z_\alpha)$, for a subsequence, one has:*

(i) $\epsilon_\alpha\to0$ *and* $\epsilon_\alpha^{2(\alpha-1)}\to\lambda$,

(ii) $v^\alpha\to\omega$ *strongly in* $H^1_{loc}(\mathbb{R}^2,\mathbb{R}^3)$ *and uniformly on compact sets of* $\mathbb{R}^2$*,*

(iii) ω *is a nonconstant solution to* $\Delta\omega=2\lambda H(\omega)\omega_x\wedge\omega_y$ *on* $\mathbb{R}^2$*.*

Notice that, according to Proposition A.1, in the limiting problem the curvature function is λH, with $\lambda\in(0,1]$, and not necessarily $\lambda=1$.

The second important result of this Appendix is variational and states a semicontinuity property, under an additional assumption on H, involving the value $\bar M_H$ defined by (2.8).

Proposition A.2 *Let $H\in C^1(\mathbb{R}^3)$ be such that $\bar M_H<1$. For $\alpha\in(1,\bar\alpha)$ let $u^\alpha\in H_0^{1,2\alpha}$ satisfy* (A.1)–(A.3)*, and let $\lambda\in(0,1]$ and $\omega\in X\cap L^\infty$ be given by Proposition A.1. Then*

$$\mathcal{E}_{\lambda H}(\omega)\le\lambda\liminf_{\alpha\to1}\mathcal{E}_H^\alpha(u^\alpha).$$

To prove Proposition A.1, first of all we need some local estimates on the family (u^α). This will be developed in Subsection A.1. Then the proof of Proposition A.1 will be performed in Subsection A.2. Finally, Proposition A.2 will be proved in Subsection A.3.

A.2. *Local estimates (ε-regularity)*

Here we study the regularity properties of critical points for $\mathcal{E}_H^\alpha$, following the arguments by Sacks and Uhlenbeck [15].

The first (minor) difference with respect to the framework of Sacks and Uhlenbeck paper lies in the nonlinear term. In [15] the Euler-Lagrange equation for the harmonic map problem involves the second fundamental form of the embedding of the target space N into an Euclidean space, instead of the curvature term. This is far to lead to any extra difficulty, since the invariance of the curvature term with respect to dilations makes computations even easier, in this case.

The main difference with [15] concerns the L^∞ bound on the maps u under consideration. In their paper, Sacks and Uhlenbeck deal with maps u whose target

space is a compact Riemannian manifold, and therefore, they have a natural L^∞ bound on all maps u. On the contrary, the target space of our maps u is the noncompact space $\mathbb{R}^3$, and hence we have no natural *a priori* bound. Therefore we have to ask it as an hypothesis.

Another difference with respect to the proof of Sacks and Uhlenbeck is due to the presence of a boundary in the domain. However, this does not lead any extra difficulty. One can argue, for example, as in Struwe [19], Proposition 2.6.

The first result concerns global regularity for fixed $\alpha > 1$, and it can be obtained as in [15], using Theorem 1.11.1′ in [14] and Struwe [19], proof of Proposition 2.6, for the regularity up to the boundary.

Lemma A.3 *Let $H \in C^1(\mathbb{R}^3)$ and let $u \in H_0^{1,2\alpha}$ be a critical point of $\mathcal{E}_H^\alpha$ for some $\alpha > 1$. Then u belongs to $W^{2,q}(D,\mathbb{R}^3)$ for every $q \in [1,+\infty)$ and solves*

$$\Delta u = -\frac{2(\alpha-1)}{1+|\nabla u|^2}(\nabla^2 u, \nabla u)\nabla u + \frac{2H(u)}{(1+|\nabla u|^2)^{\alpha-1}} u_x \wedge u_y \quad \text{in } D\ . \tag{A.4}$$

The second result of this Section concerns some local estimates for the solutions of the approximating problems (ε-*regularity*) which are actually the same as in the celebrated paper [15], and which are stated in the following Lemma (compare also with Lemma A.1 in [2]). We restrict ourselves to make estimates in the interior of the disk, thanks to the extension argument by Struwe [19].

Lemma A.4 (Main Estimate) *Let $H \in C^1(\mathbb{R}^3) \cap L^\infty$. Then there exist $\bar\varepsilon = \bar\varepsilon(\|H\|_\infty) > 0$, and for every $p \in (1,+\infty)$ an exponent $\alpha_p > 1$ and a constant $C_p = C_p(\|H\|_\infty) > 0$, such that if $\alpha \in [1,\alpha_p)$ and $u \in W^{2,p}_{loc}(D,\mathbb{R}^3)$ solves* (A.4), *then*

$$\|\nabla u\|_{L^2(D_R(z))} \le \bar\varepsilon \ \Rightarrow\ \|\nabla u\|_{H^{1,p}(D_{R/2}(z))} \le C_p R^{\frac{2}{p}-2} \|\nabla u\|_{L^2(D_R(z))}$$

for every disc $\overline{D_R(z)} \subset D$.

Proof. Our arguments strictly follow the original proof in [15]. Let u be a solution to (A.4) for some $\alpha \ge 1$. Fixing $z \in D$, for $R \in (0, 1-|z|)$ we expand $D_R(z)$ to the unit disc D, and we define a map $\omega\colon D \to \mathbb{R}^3$ by setting

$$\omega(\zeta) = u(R\zeta + z) - \fint_{D_R(z)} u$$

A direct computation shows that ω is a regular solution in D to the system

$$\Delta\omega = -\frac{2(\alpha-1)}{R^2+|\nabla\omega|^2}(\nabla^2\omega, \nabla\omega)\nabla\omega + 2\frac{H_R(\omega)}{(R^2+|\nabla\omega|^2)^{\alpha-1}}\omega_x \wedge \omega_y \tag{A.5}$$

where $H_R(\omega) = R^{2(\alpha-1)}H(\omega + \fint_{D_R(z)} u)$. Note also that $R^{-2(\alpha-1)}\|H_R\|_\infty \le \overline{H} =: \|H\|_\infty$. Now fix four radii $\frac{1}{2} = r_0 < r_1 < r_2 < r_3 = 1$ and three cut-off functions $\varphi_i \in C^\infty(\mathbb{R}^2,[0,1])$ such that $\varphi_i \equiv 1$ on $D_{r_{i-1}}$, $\varphi_i \equiv 0$ on $\mathbb{R}^2 \setminus D_{r_i}$ $(i = 1,2,3)$.

Let $K = \max_i \left(\|\nabla\varphi_i\|_\infty + \|\nabla^2\varphi_i\|_\infty\right)$. Our aim is to use equation (A.5) in order to obtain some estimate on $\varphi_i\omega$. First, we point out some simple inequalities:

$$|\Delta(\varphi_i\omega)| \le \varphi_i|\Delta\omega| + 2|\nabla\varphi_i|\ |\nabla\omega| + |\Delta\varphi_i|\ |\omega|\ , \tag{A.6}$$

and

$$\left|\frac{\varphi_i(\nabla^2\omega,\nabla\omega)\nabla\omega}{R^2+|\nabla\omega|^2}\right| \le |\varphi_i\nabla^2\omega| \le |\nabla^2(\varphi_i\omega)| + 2|\nabla\varphi_i|\ |\nabla\omega| + |\nabla^2\varphi_i|\ |\omega|\ . \tag{A.7}$$

In order to handle the curvature term in (A.5) we observe that $2\varphi_i(\omega_x\wedge\omega_y) = [(\varphi_i\omega)_x\wedge\omega_y + \omega_x\wedge(\varphi_i\omega)_y] - [(\varphi_i)_x(\omega\wedge\omega_y) + (\varphi_i)_y(\omega_x\wedge\omega)]$ and hence $|2\varphi_i(\omega_x\wedge\omega_y)| \le 2|\nabla(\varphi_i\omega)|\ |\nabla\omega| + |\nabla\varphi_i|\ |\omega|\ |\nabla\omega|$. Therefore, we can estimate

$$\left|\frac{2\varphi_i H_R(\omega)\omega_x\wedge\omega_y}{(R^2+|\nabla\omega|^2)^{\alpha-1}}\right| \le 2\overline{H}|\nabla(\varphi_i\omega)|\ |\nabla\omega| + \overline{H}|\nabla\varphi_i|\ |\omega|\ |\nabla\omega|\ . \tag{A.8}$$

Multiplying (A.5) by φ_i and using (A.6)–(A.8) we obtain

$$\begin{aligned} |\Delta(\varphi_i\omega)| \quad &\le \quad 2(\alpha-1)|\nabla^2(\varphi_i\omega)| + 6K\chi_i(|\omega| + |\nabla\omega|) \\ &\quad +2\overline{H}\ |\nabla(\varphi_i\omega)|\ |\nabla\omega| + \overline{H}\ |\nabla\varphi_i|\ |\omega|\ |\nabla\omega| \end{aligned}$$

where χ_i is the characteristic function of the set D_{r_i}. Thus, for all $p\in(1,+\infty)$ we have

$$\begin{aligned} \|\Delta(\varphi_i\omega)\|_{L^p(D_{r_i})} \ &\le 2(\alpha-1)\|\varphi_i\omega\|_{H^{2,p}(D_{r_i})} + 6K\left(\|\omega\|_{L^p(D_{r_i})} + \|\nabla\omega\|_{L^p(D_{r_i})}\right) \\ &\quad + 2\overline{H}\|\ |\nabla(\varphi_i\omega)|\ |\nabla\omega|\ \|_{L^p(D_{r_i})} + \overline{H}K\|\ |\omega|\ |\nabla\omega|\ \|_{L^p(D_{r_i})}. \end{aligned} \tag{A.9}$$

Since ω has zero mean value on D, we have that for every $p\in(1,+\infty)$

$$\|\omega\|_{L^p(D_{r_i})} \le C_p\|\nabla\omega\|_{L^2(D)}\ , \tag{A.10}$$

where C_p depends only on the Sobolev embedding constant of $H^{1,2}(D)$ into $L^p(D)$ and on the Poincaré constant on D. Taking $p\in(1,2]$, we plainly have

$$\|\nabla\omega\|_{L^p(D_{r_i})} \le 2\|\nabla\omega\|_{L^2(D)}\ . \tag{A.11}$$

Moreover, for $p\in(1,4)$, using Hölder inequality and (A.10), we can estimate

$$\|\ |\nabla(\varphi_i\omega)|\ |\nabla\omega|\ \|_{L^p(D_{r_i})} \le \|\nabla(\varphi_i\omega)\|_{L^4(D_{r_i})}\|\nabla\omega\|_{L^{4p/(4-p)}(D_{r_i})}\ , \tag{A.12}$$

$$\|\ |\omega|\ |\nabla\omega|\ \|_{L^p(D_{r_i})} \le C_4\|\nabla\omega\|_{L^2(D)}\|\nabla\omega\|_{L^{4p/(4-p)}(D_{r_i})}\ . \tag{A.13}$$

Now we apply the standard regularity theory for linear elliptic equations. Denoting by $c(p)$ the norm of the operator Δ^{-1} as a map from $L^p(D_{r_i})$ into $W^{2,p}\cap H^1_0(D_{r_i})$, and using (A.9)–(A.13), we obtain the following crucial inequality for $p\in(1,2]$

$$\begin{aligned} \beta_{p,\alpha}\|\varphi_i\omega\|_{H^{2,p}(D_{r_i})} \quad &\le \quad \overline{C}_p\|\nabla\omega\|_{L^2(D)} + C_4\overline{H}K\|\nabla\omega\|_{L^2(D)}\|\nabla\omega\|_{L^{4p/(4-p)}(D_{r_i})} \\ &\quad +2\overline{H}\ \|\nabla(\varphi_i\omega)\|_{L^4(D_{r_i})}\|\nabla\omega\|_{L^{4p/(4-p)}(D_{r_i})}\ . \end{aligned} \tag{A.14}$$

where we have set $\beta_{p,\alpha} = c(p)^{-1} - 2(\alpha - 1)$ and $\overline{C}_p = 6K(C_p + 2)$. First, we use (A.14) taking $p = 2$ and $i = 2$. From (A.10), we have that $\|\nabla(\varphi_2\omega)\|_{L^4(D_{r_2})} \leq C_4K\|\nabla\omega\|_{L^2(D)} + \|\nabla\omega\|_{L^4(D_{r_2})}$. Then, we fix $\bar{\alpha} > 1$ such that $\beta_{2,\bar{\alpha}} > 0$, and we observe that $\bar{\alpha}$ depends only on the constants in elliptic regularity theory. Hence, if $\alpha \in [1, \bar{\alpha}]$, (A.14) with $p = 2$ and $i = 2$ yields

$$\begin{aligned} \|\omega\|_{H^{2,2}(D_{r_1})} &\leq \|\varphi_2\omega\|_{H^{2,2}(D_{r_2})} \\ &\leq C_1(\overline{H}) \left(\|\nabla\omega\|_{L^2(D)} + \|\nabla\omega\|_{L^4(D_{r_2})}\|\nabla\omega\|_{L^2(D)} + \|\nabla\omega\|^2_{L^4(D_{r_2})} \right), \end{aligned} \tag{A.15}$$

where $C_1(\overline{H})$ depends only on $\overline{H}$. Now we show that $\|\nabla\omega\|_{L^4(D_{r_2})}$ can be controlled in terms of $\|\nabla\omega\|_{L^2(D)}$, if $\|\nabla\omega\|_{L^2(D)}$ is small enough. To do this, we use again (A.14) taking $p = \frac{4}{3}$ and $i = 3$. We point out that the critical Sobolev exponent corresponding to $p = \frac{4}{3}$ is $p^* = 4$. Hence, there exists $S_{4/3} > 0$ (independent of the domain) such that $S_{4/3}\|\varphi_3\omega\|_{H^{1,4}(D)} \leq \|\varphi_3\omega\|_{H^{2,4/3}(D)}$. Therefore, reminding that $r_3 = 1$, (A.14), with $p = \frac{4}{3}$ and $i = 3$, yields

$$\left(\beta_{\frac{4}{3},\alpha} - 2\overline{H}S^{-1}_{4/3}\|\nabla\omega\|_{L^2(D)} \right) \|\varphi_3\omega\|_{H^{2,4/3}(D)} \leq \overline{C}_{4/3}\|\nabla\omega\|_{L^2(D)} + C_4\overline{H}K\|\nabla\omega\|^2_{L^2(D)}.$$

Now, take a smaller $\bar{\alpha} > 1$ in order that $\beta_{4/3,\bar{\alpha}} = \bar{\beta} > 0$. Thus, for every $\alpha \in [1, \bar{\alpha}]$ we have $\beta_{4/3,\alpha} \geq \bar{\beta}$. Then, take $\bar{\varepsilon} > 0$ small enough, such that $\bar{\beta} - 2S^{-1}_{4/3}\overline{H}\bar{\varepsilon} > 0$. Notice that $\bar{\varepsilon}$ depends only on $\overline{H}$. Therefore, we infer that

$$\begin{aligned} \|\nabla\omega\|_{L^4(D_{r_2})} &\leq \|\nabla(\varphi_3\omega)\|_{L^4(D)} \leq \|\varphi_3\omega\|_{H^{1,4}(D)} \\ &\leq S^{-1}_{4/3}\|\varphi_3\omega\|_{H^{2,4/3}(D)} \leq C_2(\overline{H})\|\nabla\omega\|_{L^2(D)} , \end{aligned}$$

if $\|\nabla\omega\|_{L^2(D)} \leq \bar{\varepsilon}$, with $C_2(\overline{H})$ depending only on $\overline{H}$. Going back to (A.15), we have proved that

$$\|\omega\|_{H^{2,2}(D_{r_1})} \leq C_3(\overline{H})\|\nabla\omega\|_{L^2(D)}$$

when $\alpha \in [1, \bar{\alpha}]$, provided that $\|\nabla\omega\|_{L^2(D)} \leq \bar{\varepsilon}$, being $C_3(\overline{H})$ a positive constant depending only on $\overline{H}$. Hence, by the Sobolev embeddings, for every $q \in [1, +\infty)$ there exists a positive constant $C_4(q, \overline{H})$, depending also on q such that

$$\|\omega\|_{H^{1,q}(D_{r_1})} \leq C_4(q, \overline{H})\|\nabla\omega\|_{L^2(D)} \tag{A.16}$$

when $\alpha \in [1, \bar{\alpha}]$ and $\|\nabla\omega\|_{L^2(D)} \leq \bar{\varepsilon}$. For the last step, we apply (A.9) with $i = 1$ and we use the following estimates, obtained with the Hölder inequality and with (A.10):

$$\begin{aligned} \| \, |\nabla(\varphi_1\omega)| \, |\nabla\omega| \, \|_{L^p(D_{r_1})} &\leq K\| \, |\omega| \, |\nabla\omega| \, \|_{L^p(D_{r_1})} + \|\nabla\omega\|^2_{L^{2p}(D_{r_1})} , \\ \| \, |\omega| \, |\nabla\omega| \, \|_{L^p(D_{r_1})} &\leq C_{2p}\|\nabla\omega\|_{L^2(D)}\|\nabla\omega\|_{L^{2p}(D_{r_1})} . \end{aligned}$$

Then, arguing as for (A.14) we get

$$\begin{aligned} \beta_{p,\alpha}\|\varphi_1\omega\|_{H^{2,p}(D_{r_1})} &\leq 6K\|\nabla\omega\|_{L^2(D)} + 6K\|\nabla\omega\|_{L^p(D_{r_1})} \\ &\quad + 2\overline{H}\|\nabla\omega\|_{L^{2p}(D_{r_1})} + 3\overline{H}KC_{2p}\|\nabla\omega\|_{L^2(D)}\|\nabla\omega\|_{L^{2p}(D_{r_1})}. \end{aligned}$$

Finally, in order to estimate $\|\nabla\omega\|_{L^p(D_{r_1})}$ and $\|\nabla\omega\|_{L^{2p}(D_{r_1})}$, we use (A.16) with $q = p$ and $q = 2p$. Thus, for fixed $p \in (1, +\infty)$ we can find $\alpha_p \in (1, \bar{\alpha}]$ such that for $\alpha \in [1, \alpha_p]$ one has $\beta_{p,\alpha} \geq \beta_{p,\alpha_p} > 0$. Moreover, we can also find a constant $C_5(p, \overline{H}) > 0$ such that for $\alpha \in [1, \alpha_p]$, one has

$$\|\omega\|_{H^{2,p}(D_{1/2})} \leq \|\varphi_1\omega\|_{H^{2,p}(D_{r_1})} \leq C_5(p, \overline{H})\|\nabla\omega\|_{L^2(D)}$$

provided that $\|\nabla\omega\|_{L^2(D)} \leq \bar{\varepsilon}$. To conclude the proof, we just have to remark that $\|\nabla\omega\|_{L^2(D)} = \|\nabla u\|_{L^2(D_R(z))}$, and $\|\nabla u\|^p_{H^{1,p}(D_{R/2}(z))} = R^{2-p}\|\nabla\omega\|^p_{L^p(D_{1/2})} + R^{2-2p}\|\nabla^2\omega\|^p_{L^p(D_{1/2})} \leq R^{2-2p}\|\omega\|^p_{H^{2,p}(D_{1/2})}$, since $R \leq 1$. □

A.3. *Passing to the limit (blow up analysis for* (u^α)*)*

The first preliminary result concerns the behaviour of the starting sequence (u^α) satisfying (A.1)–(A.3).

Lemma A.5 $u^\alpha \to 0$ *weakly in* H^1_0 *and* $\|\nabla u^\alpha\|_\infty \to +\infty$ *as* $\alpha \to 1$.

Proof. Since (u^α) is bounded in H^1_0 and in L^∞, passing to a subsequence, we can assume that $u^\alpha \to u$ weakly in H^1_0, for some $u \in H^1_0 \cap L^\infty$. Let us prove that u is a weak solution to the Dirichlet problem

$$\begin{cases} \Delta u = 2H(u)u_x \wedge u_y & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \tag{A.17}$$

To this aim, fix an open set Ω with $\overline{\Omega} \subset D$. Arguing as in [15], proof of Proposition 4.3, we can find a finite set of points $F \subset \Omega$ such that $\int_{D_R(z)} |\nabla u|^2 \leq \bar{\varepsilon}$ for $z \notin F$ and R small enough (depending on z), where $\bar{\varepsilon} > 0$ is given by Lemma A.4. Then, an application of Lemma A.4 gives a uniform bound for $\|\nabla u^\alpha\|_{H^{1,2}(D_{R/2}(z))}$. Noting also that (u^α) is bounded in $L^q(D)$ for every $q \in [1, +\infty]$, we infer that (u^α) is bounded in $W^{2,2}(D_{R/2}(z))$, and hence, by Rellich Theorem, $u^\alpha \to u$ strongly in $H^1(D_{R/2}(z))$ and in $L^\infty(D_{R/2}(z))$. This is sufficient to conclude that u is a weak solution to the equation $\Delta u = 2H(u)u_x \wedge u_y$ in $D_{R/2}(z)$ and hence, since z was arbitrarily chosen, in $\Omega \setminus F$. Now we can repeat the proof of Theorem 3.6 in [15]. Assume for simplicity that $F = \{0\}$. Let $\eta \in C^\infty(\mathbb{R}, [0, 1])$ be such that $\eta(s) = 0$ for $s \leq 1$ and $\eta(s) = 1$ for $s \geq 2$, and set $\eta^k(s) = \eta(ks)$. Given $h \in C^\infty_c(\Omega, \mathbb{R}^3)$ we set $h^k(\zeta) = \eta^k(|\zeta|)h(\zeta)$. Notice that h^k can be used as test for u to get

$$\int_\Omega \nabla u \cdot \nabla h^k + 2\int_\Omega H(u)h^k \cdot u_x \wedge u_y = 0\ . \tag{A.18}$$

Now, since $h^k \to h$ weakly* in L^∞, we get $\int_\Omega H(u)h^k \cdot u_x \wedge u_y \to \int_\Omega H(u)h \cdot u_x \wedge u_y$. Also, $\int_\Omega \nabla u \cdot \nabla h^k \to \int_\Omega \nabla u \cdot \nabla h$, since, by Hölder inequality, $\int_\Omega |\nabla u \cdot \nabla\eta^k||h| \leq C\|\nabla u\|_{L^2(D_{2/k})} = o(1)$ as $k \to +\infty$. Therefore, (A.18) yields in the limit

$$\int_\Omega \nabla u \cdot \nabla h + 2\int_\Omega H(u)h \cdot u_x \wedge u_y = 0$$

for every test function $h \in C_c^\infty(\Omega, \mathbb{R}^3)$, that is, u solves $\Delta u = 2H(u) u_x \wedge u_y$ in Ω. Finally, for the arbitrariness of Ω, we conclude that u is a weak solution to problem (A.17). Then, by a Heinz regularity result [9], u is smooth, and a nonexistence result by Wente [21], which holds also in case H nonconstant, can be applied, to conclude that $u \equiv 0$. Thus, we obtain that $u^\alpha \to 0$ weakly in H_0^1 and strongly in $H^1_{loc}(D \setminus N)$ where N is a countable set of D. In particular $\nabla u^\alpha \to 0$ pointwise a.e. in D. Therefore, as a last step, we observe that if it were $\liminf_{\alpha \to 1} \|\nabla u^\alpha\|_\infty < +\infty$, then $\liminf_{\alpha \to 1} \|\nabla u^\alpha\|_2 = 0$, contrary to (A.3). Hence, it must be $\|\nabla u^\alpha\|_\infty \to +\infty$ as $\alpha \to 1$. □

Proof of Proposition A.1. For every $\alpha \in (1, \bar\alpha)$ set $\epsilon_\alpha = \|\nabla u^\alpha\|_\infty^{-1}$, let $z_\alpha \in \overline{D}$ be such that $|\nabla u^\alpha(z_\alpha)| = \epsilon_\alpha^{-1}$ and define

$$v^\alpha(z) = u^\alpha(\epsilon_\alpha z + z_\alpha) \ . \tag{A.19}$$

Note that $v^\alpha \in H_0^1(D_\alpha, \mathbb{R}^3)$ where $D_\alpha = D_{\epsilon_\alpha^{-1}}\big(-\frac{z_\alpha}{\epsilon_\alpha}\big)$. Moreover the following facts hold:

$$\|v^\alpha\|_\infty = \|u^\alpha\|_\infty \tag{A.20}$$

$$\|\nabla v^\alpha\|_2 = \|\nabla u^\alpha\|_2 \tag{A.21}$$

$$|\nabla v^\alpha(0)| = \|\nabla v^\alpha\|_\infty = 1 \ . \tag{A.22}$$

Furthermore, $v^\alpha \in W^{2,q}_{loc}(D_\alpha, \mathbb{R}^3)$ for every $q \in [1, +\infty)$ and solves the system

$$\Delta v^\alpha = -\frac{2(\alpha-1)}{\epsilon_\alpha^2 + |\nabla v^\alpha|^2}(\nabla^2 v^\alpha, \nabla v^\alpha)\nabla v^\alpha + \frac{2\epsilon_\alpha^{2(\alpha-1)} H(v^\alpha)}{(\epsilon_\alpha^2 + |\nabla v^\alpha|^2)^{\alpha-1}} v_x^\alpha \wedge v_y^\alpha \ \text{ in } D_\alpha \ . \tag{A.23}$$

Since $\epsilon_\alpha \to 0$ as $\alpha \to 1$, one has that $0 < \epsilon_\alpha^{2(\alpha-1)} < 1$, and then, for a subsequence, $\epsilon_\alpha^{2(\alpha-1)} \to \lambda$ for some $\lambda \in [0,1]$. Moreover, setting $\rho_\alpha = \epsilon_\alpha^{-1} \mathrm{dist}(z_\alpha, \partial D)$, we may also assume that there exists $\lim_{\alpha \to 1} \rho_\alpha \in [0, +\infty]$. Let Ω_∞ be the union of all compact sets in $\mathbb{R}^2$ contained in D_α as $\alpha \to 1$. Note that Ω_∞ is a half-plane if $\rho_\alpha \to \ell \in [0, +\infty)$, while $\Omega_\infty = \mathbb{R}^2$ if $\rho_\alpha \to +\infty$. From (A.2), (A.20) and (A.21) it follows that there exists $\omega \in X \cap L^\infty$ such that, for a subsequence, $\nabla v^\alpha \to \nabla \omega$ weakly in $(L^2(\mathbb{R}^2, \mathbb{R}^3))^2$. Moreover, by (A.22) one has that $v^\alpha \to \omega$ strongly in $L^\infty_{loc}(\mathbb{R}^2, \mathbb{R}^3)$. Let $\bar\varepsilon > 0$ be given by Lemma A.4. Take an arbitrary compact set K in Ω_∞ and set $R_K = \mathrm{dist}(K, \partial\Omega_\infty)$. Then, let $R \in (0, \min\{1, R_K, \frac{\bar\varepsilon}{\sqrt{\pi}}\})$. Hence, there exists $\alpha_K > 1$ such that $K \subset D_\alpha$ for $\alpha \in (1, \alpha_K)$ and, consequently, for every $z \in K$, one has $\overline{D_R(z)} \subset D_\alpha$ and $\|\nabla v^\alpha\|_2 \le \bar\varepsilon$. Because of the definition (A.19) of v^α, one can apply Lemma A.4, in order to conclude that $\|\nabla v^\alpha\|_{H^{1,p}(D_{R/2}(z))}$ is uniformly bounded with respect to $\alpha \in (1, \alpha_K)$, for every $p > 1$. Using (A.20) and (A.2), we infer that (v^α) is bounded in $H^{2,p}(D_{R/2}(z))$. Therefore we can conclude that $\omega \in H^{2,p}(D_{R/2}(z))$, $v^\alpha \to \omega$ strongly in $H^1(D_{R/2}(z))$, and $\nabla v^\alpha \to \nabla\omega$ pointwise everywhere in $D_{R/2}(z)$. Since z is an arbitrary point in K and K is any compact set in Ω_∞, a standard diagonal argument yields that $\omega \in H^{2,p}_{loc}(\Omega_\infty)$ for every $p < +\infty$,

$v^\alpha \to \omega$ strongly in $H^1_{loc}(\Omega_\infty)$, and $\nabla v^\alpha \to \nabla\omega$ pointwise everywhere in $\mathbb{R}^2$. In particular, by (A.22), $|\nabla\omega(0)| = \|\nabla\omega\|_\infty = 1$, and thus ω is nonconstant. Now we test (A.23) on an arbitrary function $h \in C^\infty_c(D_{R/2}(z), \mathbb{R}^3)$ and we pass to the limit as $\alpha \to 1$. First, we have

$$\int_{\mathbb{R}^2} \Delta v^\alpha \cdot h \to \int_{\mathbb{R}^2} \nabla\omega \cdot \nabla h\ , \tag{A.24}$$

because of the weak convergence $\nabla v^\alpha \to \nabla\omega$. Secondly, using the estimate

$$\left| \int_{\mathbb{R}^2} \frac{(\nabla^2 v^\alpha, \nabla v^\alpha)\nabla v^\alpha \cdot h}{\epsilon_\alpha^2 + |\nabla v^\alpha|^2} \right| \le \int_{\mathbb{R}^2} |\nabla^2 v^\alpha|\ |h| \le \|\nabla^2 v^\alpha\|_{L^p(D_{R/2}(z))} \|h\|_{L^{p'}}$$

and the fact that v^α is uniformly bounded in $H^{2,p}(D_{R/2}(z))$ as $\alpha \in (1, \alpha_K)$, we obtain that

$$2(\alpha - 1)\int_{\mathbb{R}^2} \frac{(\nabla^2 v^\alpha, \nabla v^\alpha)\nabla v^\alpha \cdot h}{\epsilon_\alpha^2 + |\nabla v^\alpha|^2} \to 0, \tag{A.25}$$

as $\alpha \to 1$. Lastly, setting

$$w^\alpha = \epsilon_\alpha^{2(\alpha-1)} \left(\frac{1}{(\epsilon_\alpha^2 + |\nabla v^\alpha|^2)^{\alpha-1}} - 1 \right) v^\alpha_x \wedge v^\alpha_y$$

one has

$$\int_{\mathbb{R}^2} \frac{\epsilon_\alpha^{2(\alpha-1)} H(v^\alpha)}{(\epsilon_\alpha^2 + |\nabla v^\alpha|^2)^{\alpha-1}} h \cdot v^\alpha_x \wedge v^\alpha_y = \int_{\mathbb{R}^2} H(v^\alpha) h \cdot w^\alpha + \epsilon_\alpha^{2(\alpha-1)} \int_{\mathbb{R}^2} H(v^\alpha) h \cdot v^\alpha_x \wedge v^\alpha_y\ .$$

Since $\epsilon_\alpha^{2(\alpha-1)} \to \lambda$, $H(v^\alpha) \to H(\omega)$ uniformly on $\overline{D_{R/2}(z)}$ and $\nabla v^\alpha \to \nabla\omega$ pointwise in $D_{R/2}(z)$, by (A.22), on one hand we infer that

$$\epsilon_\alpha^{2(\alpha-1)} \int_{\mathbb{R}^2} H(v^\alpha) h \cdot v^\alpha_x \wedge v^\alpha_y \to \lambda \int_{\mathbb{R}^2} H(\omega) h \cdot \omega_x \wedge \omega_y\ .$$

On the other hand, since $\epsilon_\alpha \in (0,1)$, we observe that

$$|w^\alpha| \le (1 + \epsilon_\alpha^{2(\alpha-1)}) |v^\alpha_x|\ |v^\alpha_y| \le |\nabla v^\alpha|^2 \le 1$$

and $w^\alpha(\zeta) \to 0$ for every $\zeta \in D_{R/2}(z)$. Indeed, if $\nabla\omega(\zeta) = 0$ then $|w^\alpha| \le |\nabla v^\alpha|^2 \to 0$, while if $\nabla\omega(\zeta) \ne 0$ then $(\epsilon_\alpha^2 + |\nabla v^\alpha(\zeta)|^2)^{\alpha-1} \to 0$. In conclusion, by the dominated convergence Theorem, we obtain that $\int_{\mathbb{R}^2} H(v^\alpha) h \cdot w^\alpha \to 0$ and then

$$\int_{\mathbb{R}^2} \frac{2\epsilon_\alpha^{2(\alpha-1)} H(v^\alpha)}{(\epsilon_\alpha^2 + |\nabla v^\alpha|^2)^{\alpha-1}} h \cdot v^\alpha_x \wedge v^\alpha_y \to 2\lambda \int_{\mathbb{R}^2} H(\omega) h \cdot \omega_x \wedge \omega_y\ , \tag{A.26}$$

as $\alpha \to 1$. Then (A.23)–(A.26) imply that

$$\int_{\mathbb{R}^2} \nabla\omega \cdot \nabla h + 2\lambda \int_{\mathbb{R}^2} H(\omega) h \cdot \omega_x \wedge \omega_y = 0$$

for every $h \in C_c^\infty(D_{R/2}(z), \mathbb{R}^3)$, for every $z \in K$ and for every compact set K in Ω_∞, that is, ω solves $\Delta\omega = 2\lambda H(\omega)\omega_x \wedge \omega_y$ in Ω_∞. Suppose that Ω_∞ is a half-plane. Since $v^\alpha = 0$ on ∂D_α, one has that $\omega = 0$ on $\partial\Omega_\infty$. Moreover, since a half-plane is conformally equivalent to a disc, ω gives arise to a nonconstant solution to the Dirichlet problem

$$\begin{cases} \Delta u = 2\lambda H(u) u_x \wedge u_y & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \tag{A.27}$$

As already noted in the proof of Lemma A.5, the only solution to (A.27) is $u \equiv 0$, and this gives a contradiction, since ω is nonconstant. Hence, it must be $\Omega_\infty = \mathbb{R}^2$, that is, ω is a λH-bubble. Finally, we observe that $\lambda > 0$, since the only bounded solutions to $\Delta u = 0$ on $\mathbb{R}^2$ with $\mathcal{D}(u) < +\infty$ are the constant functions, and we already know that ω is nonconstant. This concludes the proof. □

A.4. *Proof of Proposition A.2*

For every domain Ω in $\mathbb{R}^2$, $\alpha \in (1, \bar{\alpha})$, and $\lambda \in (0, 1]$, set

$$\begin{aligned} \tilde{\mathcal{E}}_H^\alpha(v^\alpha, \Omega) &= \frac{1}{2\alpha}\int_\Omega \left((\epsilon_\alpha^2 + |\nabla v^\alpha|^2)^\alpha - \epsilon_\alpha^{2\alpha}\right) + 2\epsilon_\alpha^{2(\alpha-1)}\int_\Omega m_H(v^\alpha) v^\alpha \cdot v_x^\alpha \wedge v_y^\alpha \\ \mathcal{E}_{\lambda H}(\omega, \Omega) &= \frac{1}{2}\int_\Omega |\nabla\omega|^2 + 2\lambda\int_\Omega m_H(\omega)\omega \cdot \omega_x \wedge \omega_y \ . \end{aligned}$$

Notice that $\tilde{\mathcal{E}}_H^\alpha(v^\alpha, D_\alpha) = \epsilon_\alpha^{2(\alpha-1)}\mathcal{E}_H^\alpha(u^\alpha)$ and v^α solves the system

$$\operatorname{div}\left((\epsilon_\alpha^2 + |\nabla v^\alpha|^2)^{\alpha-1}\nabla v^\alpha\right) = 2\epsilon_\alpha^{2(\alpha-1)} H(v^\alpha) v_x^\alpha \wedge v_y^\alpha \ . \tag{A.28}$$

Now, multiplying (A.28) by v^α, we obtain

$$\begin{aligned} \operatorname{div}\left((\epsilon_\alpha^2 + |\nabla v^\alpha|^2)^{\alpha-1}\nabla v^\alpha \cdot v^\alpha\right) &= (\epsilon_\alpha^2 + |\nabla v^\alpha|^2)^{\alpha-1}|\nabla v^\alpha|^2 \\ &\quad + 2\epsilon_\alpha^{2(\alpha-1)} H(v^\alpha) v^\alpha \cdot v_x^\alpha \wedge v_y^\alpha \ . \end{aligned} \tag{A.29}$$

Integrating (A.29) on a domain Ω and using the divergence theorem we infer that

$$\begin{aligned} \int_{\partial\Omega} (\epsilon_\alpha^2 + |\nabla v^\alpha|^2)^{\alpha-1} v^\alpha \cdot \frac{\partial v^\alpha}{\partial\nu} &= \int_\Omega (\epsilon_\alpha^2 + |\nabla v^\alpha|^2)^{\alpha-1}|\nabla v^\alpha|^2 \\ &\quad + 2\epsilon_\alpha^{2(\alpha-1)}\int_\Omega H(v^\alpha) v^\alpha \cdot v_x^\alpha \wedge v_y^\alpha \ . \end{aligned} \tag{A.30}$$

Using (2.8) and the definition of $\tilde{\mathcal{E}}_H^\alpha(v^\alpha, \Omega)$ we can estimate

$$\begin{aligned} 2\epsilon_\alpha^{2(\alpha-1)}\int_\Omega H(v^\alpha) v^\alpha \cdot v_x^\alpha \wedge v_y^\alpha &\leq \epsilon_\alpha^{2(\alpha-1)}\frac{\bar{M}_H}{2}\int_\Omega |\nabla v^\alpha|^2 + 3\tilde{\mathcal{E}}_H^\alpha(v^\alpha, \Omega) \\ &\quad - \frac{3}{2\alpha}\int_\Omega \left((\epsilon_\alpha^2 + |\nabla v^\alpha|^2)^\alpha - \epsilon_\alpha^{2\alpha}\right). \end{aligned} \tag{A.31}$$

Hence, setting

$$\begin{aligned}
I_\alpha(\partial\Omega) &= \frac{1}{3}\int_{\partial\Omega}(\epsilon_\alpha^2+|\nabla v^\alpha|^2)^{\alpha-1}v^\alpha\cdot\frac{\partial v^\alpha}{\partial\nu}\\
I_\alpha(\Omega) &= \frac{1}{2\alpha}\int_\Omega\left((\epsilon_\alpha^2+|\nabla v^\alpha|^2)^\alpha-\epsilon_\alpha^{2\alpha}\right)-\frac{1}{3}\int_\Omega(\epsilon_\alpha^2+|\nabla v^\alpha|^2)^{\alpha-1}|\nabla v^\alpha|^2\\
&\quad -\epsilon_\alpha^{2(\alpha-1)}\frac{\bar{M}_H}{6}\int_\Omega|\nabla v^\alpha|^2,
\end{aligned}$$

by (A.31) the equation (A.30) becomes

$$\tilde{\mathcal{E}}_H^\alpha(v^\alpha,\Omega)\ge I_\alpha(\partial\Omega)+I_\alpha(\Omega)\ . \tag{A.32}$$

With algebraic computations, one has

$$\begin{aligned}
I_\alpha(\Omega) &\ge \left(\frac{1}{2\alpha}-\frac{1}{3}\right)\int_\Omega\left((\epsilon_\alpha^2+|\nabla v^\alpha|^2)^\alpha-\epsilon_\alpha^{2\alpha}\right)-\epsilon_\alpha^{2(\alpha-1)}\frac{\bar{M}_H}{6}\int_\Omega|\nabla v^\alpha|^2\\
&\ge \epsilon_\alpha^{2(\alpha-1)}\left(\frac{1}{2}-\frac{\alpha}{3}-\frac{\bar{M}_H}{6}\right)\int_\Omega|\nabla v^\alpha|^2\ .
\end{aligned}$$

Then, since $\bar{M}_H<1$, one obtains that $I_\alpha(\Omega)\ge 0$ for $\alpha>1$ sufficiently close to 1, whatever Ω is. Hence, (A.32) reduces to

$$\tilde{\mathcal{E}}_H^\alpha(v^\alpha,\Omega)\ge I_\alpha(\partial\Omega)\ . \tag{A.33}$$

Now we take $\Omega=\mathbb{R}^2\setminus D_R$. First, we observe that, since $v^\alpha\to\omega$ strongly in $H^1_{loc}(\mathbb{R}^2,\mathbb{R}^3)$ and uniformly on compact sets, and $\epsilon_\alpha^{2(\alpha-1)}\to\lambda$, it holds that

$$\begin{aligned}
\lim_{\alpha\to1}\tilde{\mathcal{E}}_H^\alpha(v^\alpha,D_R) &= \mathcal{E}_{\lambda H}(\omega,D_R)\\
\limsup_{\alpha\to1}|I_\alpha(\partial D_R)| &\le \frac{1}{3}\left|\int_{\partial D_R}\omega\cdot\frac{\partial\omega}{\partial\nu}\right|
\end{aligned}$$

for every $R>0$. Then, by (A.33), we obtain

$$\begin{aligned}
\mathcal{E}_{\lambda H}(\omega,D_R) &= \tilde{\mathcal{E}}_H^\alpha(v^\alpha)-\tilde{\mathcal{E}}_H^\alpha(v^\alpha,\mathbb{R}^2\setminus D_R)+o(1)\\
&\le \epsilon_\alpha^{2(\alpha-1)}\mathcal{E}_H^\alpha(u^\alpha)+\frac{1}{3}\left|\int_{\partial D_R}\omega\cdot\frac{\partial\omega}{\partial\nu}\right|+o(1)
\end{aligned}$$

where $o(1)\to0$ as $\alpha\to1$, for every $R>0$. Hence

$$\lambda\liminf_{\alpha\to1}\mathcal{E}_H^\alpha(u^\alpha)\ge\mathcal{E}_{\lambda H}(\omega,D_R)-\frac{1}{3}\left|\int_{\partial D_R}\omega\cdot\frac{\partial\omega}{\partial\nu}\right| \tag{A.34}$$

for every $R>0$. Finally, notice that

$$\left|\int_{\partial D_R}\omega\cdot\frac{\partial\omega}{\partial\nu}\right| = \left|\int_{\mathbb{R}^2\setminus D_R}\left(\omega\cdot\Delta\omega+|\nabla\omega|^2\right)\right|$$

$$
\begin{aligned}
&= \left| \int_{\mathbb{R}^2\setminus D_R} \left(2\lambda H(\omega)\omega \cdot \omega_x \wedge \omega_y + |\nabla\omega|^2\right) \right| \\
&\leq (\lambda \|H\|_\infty \|\omega\|_\infty + 1) \int_{\mathbb{R}^2\setminus D_R} |\nabla\omega|^2.
\end{aligned}
$$

Then, passing to the limit as $R \to +\infty$, from (A.34) the thesis follows. □

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